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A Kinematic Theory of Large Magnetic Reynolds Number Dynamos

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A KINEMATIC THEORY OF LARGE MAGNETIC REYNOLDS NUMBER DYNAMOS

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CONTENTS

	PAGE
1. Introduction	432
2. The basic equations	439
(*a*) Some preliminary identities	439
(*b*) The transformation of the magnetic induction equation	441
3. α^2-Dynamos	446
4. $\alpha\omega$-Dynamos	449
5. The hydromagnetic dynamo	453
6. The coordinate transformation	456
7. Discussion	458
Appendix A	459
Appendix B	460
Appendix C	460
References	462

An asymptotic analysis is made of the magnetic induction equation for certain flows characterized by a large magnetic Reynolds number R. A novel feature is the hybrid approach given to the problem. Advantage is taken of a combination of Eulerian and Lagrange coordinates. Under certain conditions the problem can be reduced to solving a pair of coupled partial differential equations dependent on only two space coordinates (cf. Braginskiĭ 1964*a*). Two main cases are considered. First the case is examined, in which the production of azimuthal magnetic field from the meridional magnetic field by a shear in the aximuthal flow is negligible. It is shown that a term $\boldsymbol{J}$ (analogous to electric current) is related linearly to the vector $\boldsymbol{B}$ which determines the magnetic field. (Note that $\boldsymbol{B}$ is not the magnetic field vector: see (1.33) and (2.35*b*).) The current $\boldsymbol{J}$ is likely to sustain dynamo action. Secondly, the case is considered, in which shearing of meridional magnetic field is the principal mechanism for creating the azimuthal magnetic field and the effect described above is one mechanism for creating meridional magnetic field from the azimuthal magnetic field. It is shown that the term $\boldsymbol{J}$ is not only linearly related to $\boldsymbol{B}$, but has an additional contribution $\boldsymbol{P}\times(\nabla\times\boldsymbol{B})$, where $\boldsymbol{P}$ is characterized by the flow (see (4.15)). Both these effects have been predicted previously in theories of dynamo action produced by turbulent motions. Under certain restrictive conditions the resulting equations in the second case reduce to Braginskiĭ's (1964*a*, *b*) formulation for nearly symmetric dynamos. The words azimuthal and meridional are not used here in the usual sense. The difference in terminology is a consequence of a coordinate transformation.

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1. Introduction

During recent years several models have been proposed to explain geomagnetic and stellar dynamos. Three main types have had considerable success; statistical, multiple length scale and nearly symmetric dynamo models. All three approaches have one feature in common, namely to introduce a modified Ohm's law by explicit (or implicit) averaging. The usual form of Ohm's law in an electrically conducting fluid is

$$\boldsymbol{j}^* = \sigma(\boldsymbol{E}^* + \boldsymbol{u}^* \times \boldsymbol{b}^*), \tag{1.1}$$

where $\boldsymbol{j}^*$ is the electric current, $\boldsymbol{E}^*$ is the electric field, $\boldsymbol{b}^*$ is the magnetic field, σ is the electrical conductivity and $\boldsymbol{u}^*$ is the fluid velocity. It is well known that dynamo models cannot be found which satisfy (1.1) and Maxwell's equations if the fluid velocity and magnetic field contain too much symmetry. Indeed Cowling (1933) showed that steady dynamos could not exist with axisymmetric magnetic fields. Such considerations have hindered the search for dynamo models.

The statistical approach has recently received considerable attention. Since many of the new terms introduced into (1.1) after statistical averaging have their counterpart in the theory of well-ordered motions presented in this paper, it is worth while describing very briefly the nature of the results. The velocity $\boldsymbol{u}^*$ and magnetic field $\boldsymbol{b}^*$ are separated into their mean and fluctuating parts by setting

$$\boldsymbol{u}^* = \langle \boldsymbol{u} \rangle + \boldsymbol{u}', \quad \boldsymbol{b}^* = \langle \boldsymbol{b} \rangle + \boldsymbol{b}', \tag{1.2}$$

where the brackets represent an averaging operation and the primed vectors have zero average. At this stage it is unnecessary to give a precise definition of the averaging operation. Clearly on substituting (1.2) into (1.1) and averaging, Ohm's law becomes

$$\langle \boldsymbol{j} \rangle = \sigma\{\langle \boldsymbol{E} \rangle + \langle \boldsymbol{u} \rangle \times \langle \boldsymbol{b} \rangle + \langle \boldsymbol{u}' \times \boldsymbol{b}' \rangle\}. \tag{1.3}$$

A principal objective of the statistical theories is to determine a simple representation of $\langle \boldsymbol{u}' \times \boldsymbol{b}' \rangle$ which effectively introduces a new electromotive force in the averaged Ohm's law. Further progress can be made in determining $\langle \boldsymbol{u}' \times \boldsymbol{b}' \rangle$ if certain assumptions are justified. The type of assumptions that are made will be described later in the section. However, the approximations usually lead to the form

$$\langle \boldsymbol{u}' \times \boldsymbol{b}' \rangle_i = \alpha_{ij} \langle \boldsymbol{b} \rangle_j + \beta_{ijk} \partial \langle \boldsymbol{b} \rangle_j / \partial x_k \tag{1.4}$$

for the ith component of the vector $\langle \boldsymbol{u}' \times \boldsymbol{b}' \rangle$ in a rectangular Cartesian frame, where $\boldsymbol{x}$ is the position vector and α_{ij}, β_{ijk} are tensors which depend on the structure of the turbulent velocity. Several authors have made precise estimates of the coefficients in (1.4) (see, for example, (1.11)). For the present purpose it is sufficient to give general invariance arguments which indicate the character of the coefficients α_{ij}, β_{ijk} but cannot give their magnitude.

For the present, attention is restricted to the case where $\langle \boldsymbol{u} \rangle = 0$. If the turbulence is steady, homogeneous and isotropic, invariance arguments show that

$$\alpha_{ij} = \alpha \delta_{ij}, \quad \beta_{ijk} = \beta \epsilon_{ijk}, \tag{1.5}$$

where α and β are constants and where δ_{ij} is the unit tensor and ϵ_{ijk} is the completely anti-symmetric tensor of rank 3 with $\epsilon_{123} = 1$ for a righthanded system. It follows that Ohm's law becomes

$$\langle \boldsymbol{j} \rangle = \sigma_T\{\langle \boldsymbol{E} \rangle + \alpha \langle \boldsymbol{b} \rangle\}, \tag{1.6}$$

where

$$\sigma_T = \sigma(1 + \mu\sigma\beta)^{-1}, \tag{1.7}$$

and μ is the magnetic permeability. In accordance with the terminology of Steenbeck & Krause (1969*a*), the term $\alpha\langle \boldsymbol{b}\rangle$ in (1.6) will subsequently be referred to as the α-effect. More subtle arguments depending on the distinction between polar and axial vectors indicate that α is a pseudoscalar, e.g. the helicity $\langle \boldsymbol{u}'\cdot\nabla\times\boldsymbol{u}'\rangle$ (see Moffatt 1969) which is the scalar product of a polar and an axial vector. However, for the α to be non-zero the turbulence must lack mirror reflexional symmetry. In a cosmic body it is natural to account for the lack of symmetry by the rotation, but this introduces a preferred direction ($\boldsymbol{\lambda}$ say). In this case (1.5) is no longer valid and α_{ij}, β_{ijk} must be expressed in the form

$$\left.\begin{aligned}\alpha_{ij} &= \alpha\delta_{ij}+\alpha_0\epsilon_{ijk}\lambda_k+\gamma\lambda_i\lambda_j,\\ \beta_{ijk} &= \beta\epsilon_{ijk}-\beta_1\lambda_i\delta_{jk}-\beta_2\lambda_j\delta_{ki}-\beta_3\lambda_k\delta_{ij}+\mu_1\epsilon_{jkl}\lambda_l\lambda_i+\mu_2\epsilon_{kil}\lambda_l\lambda_j+\mu_3\epsilon_{ijl}\lambda_l\lambda_k+\nu\lambda_i\lambda_j\lambda_k,\end{aligned}\right\} \quad (1.8)$$

where α, β, μ, ν are functions of λ^2.

Attention is now fixed on a different mechanism which is introduced by the existence of a preferred direction. It can be argued that in a rotating system the small eddies have 'forgotten' the preferred direction. In this case the turbulence may be regarded as predominantly homogeneous, isotropic, and mirror symmetric but also weakly anisotropic with a preferred direction given by the local angular velocity $\langle\boldsymbol{\omega}\rangle$. Thus α_{ij} and β_{ijk} are defined by (1.8) with $\boldsymbol{\lambda}=\langle\boldsymbol{\omega}\rangle$ and terms quadratic in $\langle\boldsymbol{\omega}\rangle$ are neglected. Again arguments indicate that the pseudoscalars should vanish, namely α and α_0, noting that $\langle\boldsymbol{\omega}\rangle$ is an axial vector. For these approximations Ohm's law becomes

$$\langle \boldsymbol{j}\rangle = \sigma_T\langle \boldsymbol{E}\rangle+\beta_3\langle\boldsymbol{\omega}\rangle\times\langle\boldsymbol{j}\rangle-(\beta_2+\beta_3)\,(\nabla\langle\boldsymbol{b}\rangle)\cdot\langle\boldsymbol{\omega}\rangle. \quad (1.9)$$

Finally in the general case where α_{ij} has no special properties, the term

$$\alpha_{ij}\langle \boldsymbol{b}\rangle_j \quad (1.10)$$

that appears in Ohm's law (1.3) will also be referred to as the α-effect.

The arguments of the previous paragraphs are by no means complete. However, they serve to introduce various terms that may be important in the averaged Ohm's law (1.3). The reader referred to P. H. Roberts (1970) for a more detailed survey of the subject.

The α-effect was first introduced into a dynamo model by Parker (1955). In an axisymmetric configuration azimuthal magnetic field may be created from the meridional magnetic field by a shear in the azimuthal velocity. However, there is no corresponding mechanism for creating the meridional magnetic field. Parker envisaged that a large number of short-lived, small-scale, cyclonic convective cells superimposed on the axisymmetric motion would twist small loops from the azimuthal magnetic field lines with non-vanishing projections in the meridional plane. Supposing the field lines to be systematically twisted in a preferred direction (associated with rotation) subsequent diffusion of this magnetic field could reinforce the meridional magnetic field. In fact Parker's dynamo is an example of an $\alpha\omega$-dynamo (see P. H. Roberts (1970)) being sustained by a combination of an α-effect and shearing due to a gradient in the angular velocity (ω-effect).

More recently Steenbeck & Krause (1967) have made a direct correspondence of α with helicity. Explicit evaluation of α and β (see (1.11)) is made on the assumption that the turbulence is homogeneous and isotropic. It is further assumed that (i) $\langle \boldsymbol{u}'\cdot\boldsymbol{u}'\rangle^{\frac{1}{2}} \ll (\mu\sigma l)^{-1}$ where l is the length scale over which the fluid velocity is correlated, and where the average is an ensemble average, (ii) the time variation of the turbulence is slow, i.e. the correlation time must be large in comparison with $\mu\sigma l^2$ so that equilibrium can be established by diffusion in regions with linear dimension l. Calculations lead to

$$\alpha = -\tfrac{1}{18}\mu\sigma l^2 \langle \boldsymbol{u}' \cdot \nabla \times \boldsymbol{u}' \rangle, \quad \beta = -\tfrac{1}{18}\mu\sigma l^2 \langle \boldsymbol{u}' \cdot \boldsymbol{u}' \rangle \tag{1.11}$$

(see Steenbeck & Krause 1967, eqn (16)). Similar work has been done by Moffatt (1970*a*) on the assumption that the turbulence is stationary (independent of time) and homogeneous. It is supposed that the fluctuating quantities vary on a length scale l which is small compared to the length scale associated with the mean quantities. It is further supposed that the magnetic Reynolds number based on l and $\langle \boldsymbol{u}' \cdot \boldsymbol{u} \rangle^{\frac{1}{2}}$ is small (cf. (i) above). In this case a term of the type (1.10) is obtained (α_{ij} is symmetric) but β_{ijk} does not appear. As foreseen by the earlier remarks, Moffatt's α_{ij} becomes $\alpha\delta_{ij}$ for isotropic turbulence and moreover α is proportional to $\langle \boldsymbol{u}' \cdot \boldsymbol{\chi}' \rangle$, where $\boldsymbol{u}' = \nabla \times \boldsymbol{\chi}'$, a quantity which is a measure of the knottedness of the streamlines (Moffatt 1969; see also § 4). The absence of the β_{ijk} term can be accounted for since it is likely to result from the average of a quantity with one less spatial derivative than the knottedness and consequently negligible when compared with α_{ij}. In this connexion the multiple length scale work of Childress (1969, 1970) is also relevant. In this case well-ordered motions are considered but the generation mechanism is essentially the same as Moffatt's. Again a term of the form (1.1) appears in which α_{ij} is Hermitean (strictly this depends on the degree of the velocity field being even (see Childress 1969, 1970), but the case of degree 2 is probably usual). Since these dynamos rely solely on the α-effect for their existence they will subsequently be referred to as α^2-dynamos (see P. H. Roberts 1970).

Several explicit models have been considered which illustrate the possibility that dynamos may be sustained by the α-effect. The case of fluid contained in a sphere with α constant has been considered by Krause & Steenbeck (1967) and with α proportional to $\cos\theta$ (θ being the colatitude) by Steenbeck & Krause (1966, 1967). The multiple length scale work of Moffatt (1970*a*), Childress (1969, 1970) and G. O. Roberts (1969, 1970) also supports this possibility. Indeed the α-effect has been demonstrated experimentally by Steenbeck *et al.* (1967). In practice it is likely that the geomagnetic and stellar dynamos are $\alpha\omega$-dynamos. Again numerous models have been investigated which indicate that the $\alpha\omega$-dynamo will work (for example Parker 1955; Krause & Steenbeck 1964; Steenbeck & Krause 1966, 1967, 1969*a*, *b*). Finally with regard to (1.9) Rädler (1968*a*, *b*, 1969*a*, *b*, 1970) has shown by explicit example that the $\langle \boldsymbol{\omega} \rangle \times \langle \boldsymbol{j} \rangle$-effect is capable of creating torroidal magnetic field from poloidal magnetic field. Thus in combination with the ω-effect dynamo action is possible in the absence of the α-effect.

Though this paper is not concerned with the theory of turbulent dynamos, the extensive treatment of the dynamo models by the above authors is of considerable importance for the interpretation of the results derived here in §§ 3 and 4. The three different effects, α, ω and $\langle \boldsymbol{\omega} \rangle \times \langle \boldsymbol{j} \rangle$ all appear at some stage: the α-effect only in § 3 and all three effects in § 4. As in the case of turbulent dynamo theory, coefficients appear which are difficult to evaluate explicitly though here they are determined by precise analytical expressions. Consequently interpretation of the resulting equations is made by making simplifying assumptions about the form of the coefficients. These assumptions are comparable with those made in turbulence theory. Thus no more can be said about the possibility of dynamo action (as governed by these equations) than can be concluded by the results of the above authors. However, in view of their results it is likely that the equations derived in this paper will govern dynamo models.

Braginskiĭ (1964*a*–*c*) has had considerable success in describing the Earth's dynamo in terms of a nearly axisymmetric model, characterized by a large magnetic Reynolds number R. The dynamo equations are derived from an asymptotic analysis in which the leading terms in the expansion are the axisymmetric azimuthal velocity and magnetic field. A pair of equations (see (5.17)) are determined governing the axisymmetric part of the magnetic field. The equations are derived first on the assumption that quantities vary on a slow time scale (see § 4) and second on the assumption that quantities may also vary on a fast time scale (see § 5): in this case a time average is taken. The fast time scale is important as it corresponds to a class of magnetohydrodynamic waves that are likely to be excited in the Earth's core (see Braginskiĭ 1967, 1970*a*, *b*). It should be noted that the model is an example of an $\alpha\omega$-dynamo and a detailed account of a spherical model is given (Braginskiĭ 1964*c*).

Despite the success of Braginskiĭ's model two features make it unattractive – the most important is the lack of flexibility. Essentially a very precise ordering is required in the asymptotic expansion. The magnetic field and velocity are expanded in the form

$$\boldsymbol{u}^* = U(\rho, z)\,\boldsymbol{i}_\phi + R^{-\frac{1}{2}}\boldsymbol{u}'(\rho,\phi,z) + R^{-1}\boldsymbol{u}_{\mathrm{M}}(\rho,z), \tag{1.12}$$

where ρ, ϕ, z are cylindrical polar coordinates: z is the distance along the axis, ρ is the radial distance from the axis, ϕ is the azimuthal angle, $\boldsymbol{i}_\rho$, $\boldsymbol{i}_\phi$, $\boldsymbol{i}_z$ are the unit vectors in the ρ, ϕ, z directions respectively, the suffix M denotes meridional components and primed quantities have zero ϕ-average

$$\left.\begin{aligned} &\langle f'\rangle = 0,\\ \text{where}\quad &\langle f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f\,\mathrm{d}\phi,\\ \text{and}\quad &\langle \boldsymbol{f}\rangle = \langle f_\rho\rangle\boldsymbol{i}_\rho + \langle f_\phi\rangle\boldsymbol{i}_\phi + \langle f_z\rangle\boldsymbol{i}_z.\end{aligned}\right\} \tag{1.13}$$

Though the first term in the expansion (1.12) can be interpreted physically it is difficult to motivate the subsequent ordering except in as much as it gives rise to a dynamo model. The second unattractive feature is the extremely lengthy algebraic manipulations needed to derive very simple equations. Indeed this criticism is acknowledged by Braginskiĭ (1964*b*) where the remark is particularly pertinent to the analysis of the fast time scale. Braginskiĭ points out that a simpler derivation is likely and that such a derivation would help elucidate the mechanisms involved. Motivated by these considerations Tough (1967) extended the asymptotic analysis to the next order and showed that the simple character of the equations is retained. Much of the simplification stems from the introduction of 'effective' meridional velocity and magnetic field vectors. In a recent paper Soward (1971*b*) has discovered the real significance of these vectors. It was shown that given $U(\rho, z)$ and $\boldsymbol{u}'(\rho,\phi,z)$ an axisymmetric velocity vector $\boldsymbol{V}(\rho, z)$ could be constructed with the property that the velocity field described by the sum

$$\boldsymbol{u}_0 = U(\rho,z)\,\boldsymbol{i}_\phi + R^{-\frac{1}{2}}\boldsymbol{u}'(\rho,\phi,z) + R^{-1}\boldsymbol{V}(\rho,z)$$

has closed streamlines – all velocity vectors have zero divergence. Moreover, a velocity of the type (1.12) can be naturally decomposed into the form

$$\boldsymbol{u}^* = \boldsymbol{u}_0(\rho,\phi,z) + R^{-1}\boldsymbol{u}_{\mathrm{eM}}, \tag{1.14}$$

where, correct to lowest order, $\boldsymbol{u}_{\mathrm{eM}}$ is the 'effective' meridional velocity introduced by Braginskiĭ (1964*a*). It was also indicated how the velocity $\boldsymbol{u}_0$ might help the systematic iteration of the solution to higher order. The concept of the closed streamline flow $\boldsymbol{u}_0$ is fundamental in the present treatment of the dynamo problem.

P. H. Roberts (1970), in reviewing Braginskiĭ's work, has emphasized the importance of consistency conditions. A simple example may be derived from the steady magnetic induction equation in the dimensionless form

$$\boldsymbol{u}^* \times \boldsymbol{b}^* = \nabla \Phi + R^{-1} \nabla \times \boldsymbol{b}^*, \tag{1.15}$$

where Φ is a single valued potential. Provided $\boldsymbol{u}^*$ describes closed streamlines the magnetic field $\boldsymbol{b}^*$ must satisfy the consistency condition

$$\oint_{C(X)} \mathrm{d}\boldsymbol{X} \cdot \nabla \times \boldsymbol{b}^* = 0, \tag{1.16}$$

where $C(\boldsymbol{X})$ is the contour of a closed streamline and $\boldsymbol{X}$ is the position vector of points on the contour. However, in the large magnetic Reynolds number limit the $\nabla \times \boldsymbol{b}^*$ term is neglected in the first approximation. Thus if it is supposed that the solution of (1.15) takes the form

$$\boldsymbol{b}^* = \Theta \boldsymbol{u}^* + R^{-1} \boldsymbol{b}_1, \tag{1.17}$$

where
$$\boldsymbol{u}^* \cdot \nabla \Theta = 0 \quad (\nabla \cdot \boldsymbol{u}^* = \nabla \cdot \boldsymbol{b}^* = 0), \tag{1.18}$$

the consistency condition (1.16) gives

$$\Theta \oint_{C(X)} \frac{\mathrm{d}X}{|\boldsymbol{u}^*|} (\boldsymbol{u}^* \cdot \nabla \times \boldsymbol{u}^*) = -R^{-1} \oint_{C(X)} \mathrm{d}\boldsymbol{X} \cdot \nabla \times \boldsymbol{b}_1. \tag{1.19}$$

It follows that a solution of the type (1.17) exists only if the integral

$$\oint_{C(X)} \frac{\mathrm{d}X}{|\boldsymbol{u}^*|} (\boldsymbol{u}^* \cdot \nabla \times \boldsymbol{u}^*),$$

which is a measure of the helicity, is order R^{-1}. Such considerations are not new. Batchelor (1956) examined the Navier–Stokes equations for the limiting case in which the viscosity vanishes. In that case the flow is governed by (1.15) where $\boldsymbol{b}^*$ is the vorticity $\nabla \times \boldsymbol{u}^*$ and R is the Reynolds number.

The correspondence of arguments of the above type and the derivation of Braginskiĭ's dynamo equations is not immediately obvious. However, the distinctive feature of the Braginskiĭ expansion is the leading term for the velocity and magnetic field, namely $U(\rho, z)\, \boldsymbol{i}_\phi$ and $B(\rho, z)\, \boldsymbol{i}_\phi$ respectively. These vectors correspond to the leading terms in the expansions (1.14) and (1.17) where the averaging operation is performed along the circles ρ = const., z = const. The expansion is cumbersome owing to the existence of the order $R^{-\frac{1}{2}}$ terms. In view of the importance of the 'effective' velocity $\boldsymbol{u}_{\mathrm{eM}}$ (see (1.14)) in the resulting equations it would be more natural to take $\boldsymbol{u}_0$ for the leading term of the velocity expansion – the contours $C(\boldsymbol{X})$ would then correspond to the velocity $\boldsymbol{u}_0$. Consequently the order $R^{-\frac{1}{2}}$ terms would not appear in the expansion procedure. Of course integration along contours of a more general shape introduces new difficulties for the establishment of predictive equations. This problem is the main concern of this paper. Moreover, no restrictions are imposed on the deviation from axial symmetry.

In order to make the procedure explicit it is supposed that an electrically conducting incompressible fluid is contained in a spherical cavity. The set of loops $C(\boldsymbol{X})$ are defined by the mapping of the circles ρ = const., z = const. subject to the transformation

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t), \tag{1.20}$$

where $\boldsymbol{x}$ is the position vector (ρ, ϕ, z) on the circle, $\boldsymbol{X}$ is the corresponding point on the loop

$C(\boldsymbol{X})$ and t is the time.† The only restrictions on the transformation are that both $\boldsymbol{X}$ and $\boldsymbol{x}$ lie inside the cavity, that the first time derivative, and the first three spatial derivatives of $\boldsymbol{X}(\boldsymbol{x}, t)$ exist and that the transformation may be realized by a displacement $\boldsymbol{X}-\boldsymbol{x}$ which has zero dilatation. In this way (ρ, z) labels a loop $C(\boldsymbol{X})$ and the azimuthal angle ϕ parameterizes uniquely points on the loop. Both the points $\boldsymbol{X}$ and $\boldsymbol{x}$ play an important role in the subsequent theory. Quantities will be averaged around the loops $C(\boldsymbol{X})$. However, it is important to be able to commute the averaging operation with the differential operator. By making the point $\boldsymbol{X}$ correspond to a point on a circle, it is possible to change the complicated integral operation into a simple ϕ-integration about the circle $\rho = \text{const.}$, $z = \text{const.}$, so avoiding the difficulty. The notation is made more concise by defining $\boldsymbol{F}(\boldsymbol{x}, t)$ in cylindrical polar coordinates by the components

$$\boldsymbol{F}(\boldsymbol{x}, t) = (F_1, F_2, F_3) \equiv (F_\rho, F_\phi, F_z), \tag{1.21}$$

and by defining the differential operators

$$\nabla = (\partial_1, \partial_2, \partial_3) \equiv \left(\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\phi}, \frac{\partial}{\partial z}\right). \tag{1.22}$$

Further, if a vector is defined at $\boldsymbol{X}(\boldsymbol{x}, t)$ (such as the velocity vector $\boldsymbol{u}^*(\boldsymbol{X}, t)$), it is regarded as a function of $\boldsymbol{x}$ and so is resolved in cylindrical polar coordinates at the point $\boldsymbol{x}$. Since the transformation preserves volume it follows that

$$\delta\boldsymbol{X}^{(1)}\cdot(\delta\boldsymbol{X}^{(2)}\times\delta\boldsymbol{X}^{(3)}) = \delta\boldsymbol{x}^{(1)}\cdot(\delta\boldsymbol{x}^{(2)}\times\delta\boldsymbol{x}^{(3)}), \tag{1.23}$$

where
$$\delta\boldsymbol{X}^{(n)} = (\delta\boldsymbol{x}^{(n)}\cdot\nabla)\,\boldsymbol{X}, \quad \delta\boldsymbol{x}^{(n)} = (\mathrm{d}\rho^{(n)}, \rho\,\mathrm{d}\phi^{(n)}, \mathrm{d}z^{(n)}), \tag{1.24}$$

and $\mathrm{d}\rho^{(n)}$, $\mathrm{d}\phi^{(n)}$, $\mathrm{d}z^{(n)}$ are arbitrary differentials. However, (1.24) may also be expressed in the form

$$\delta X_i^{(n)} = a_{ij}(\boldsymbol{X})\,\delta x_j^{(n)}, \tag{1.25}$$

where
$$a_{ij}(\boldsymbol{F}) = (\partial_j \boldsymbol{F})_i = \partial_j F_i + \delta_{j2}\epsilon_{3ki}(1/\rho)\,F_k, \tag{1.26}$$

and summation is automatically assumed when a suffix is repeated.‡ Since the $\delta\boldsymbol{x}^{(n)}$ are arbitrary vectors it follows that

$$J(\boldsymbol{X}) = 1, \tag{1.27}$$

where
$$J(\boldsymbol{F}) = (1/3!)\,\epsilon_{ijk}\epsilon_{lmn}a_{il}(\boldsymbol{F})\,a_{jm}(\boldsymbol{F})\,a_{kn}(\boldsymbol{F}). \tag{1.28}$$

It will become apparent later that it is inconvenient to describe the magnetic field at $\boldsymbol{X}$ by its value there; $\boldsymbol{b}^*(\boldsymbol{X}, t)$. Instead it is determined from the related vector $\boldsymbol{b}(\boldsymbol{x}, t)$. The relation between $\boldsymbol{b}$ and $\boldsymbol{b}^*$ is best understood by interpreting $\boldsymbol{b}(\boldsymbol{x}, t)$ as a magnetic field vector. The magnetic field is supposed frozen to a (fictitious) fluid. The fluid at every point $\boldsymbol{x}$ in the cavity is subsequently moved to the new points $\boldsymbol{X}(\boldsymbol{x}, t)$ by a continuous deformation. The resultant magnetic field is $\boldsymbol{b}^*(\boldsymbol{X}, t)$. Reversing the process gives a physical interpretation of the vector $\boldsymbol{b}(\boldsymbol{x}, t)$. With this definition of $\boldsymbol{b}(\boldsymbol{x}, t)$, the flux of magnetic field across (fictitious) material surfaces is invariant under the transformation (1.20). Consequently $\boldsymbol{b}^*(\boldsymbol{X}, t)$ is related to $\boldsymbol{b}(\boldsymbol{x}, t)$ by the identity

$$\boldsymbol{b}^*(\boldsymbol{X}, t)\cdot[\delta\boldsymbol{X}^{(1)}\times\delta\boldsymbol{X}^{(2)}] = \boldsymbol{b}(\boldsymbol{x}, t)\cdot[\delta\boldsymbol{x}^{(1)}\times\delta\boldsymbol{x}^{(2)}], \tag{1.29}$$

† There is some similarity here with the use of a varying reference configuration (see, for example, Truesdell & Toupin 1960, p. 372).

‡ It should be emphasized that a_{ij} is not a tensor. Indeed the F_i defined in (1.21) are neither the covariant nor the contravariant components of a vector. Since cylindrical polar coordinates provide the only suitable reference frame it seems unnecessary to introduce the general formulation of the tensor calculus.

where $\delta X^{(n)}$ is defined by (1.24). By the same argument that leads to (1.27), it follows that

$$T_{ki}(X)\, b_k^*(X,t) = b_i(x,t), \tag{1.30}$$

where
$$T_{ij}(F) = (1/2!)\, \epsilon_{ilm}\epsilon_{jpq} a_{lp}(F)\, a_{mq}(F). \tag{1.31}$$

In view of the identities
$$a_{mi}T_{mj} = a_{im}T_{jm} = J\delta_{ij}, \tag{1.32}$$

it follows that
$$\boldsymbol{b}^*(X,t) = \boldsymbol{b}(x,t)\cdot\nabla X. \tag{1.33}$$†

The fluid velocity $\boldsymbol{u}^*$ at X is defined in almost the same way by

$$\boldsymbol{u}^*(X,t) = \partial X/\partial t + \boldsymbol{u}(x,t)\cdot\nabla X. \tag{1.34}$$

The term $\partial X/\partial t$ is the velocity vector describing the movement of the point X when x is fixed. The velocity $\boldsymbol{u}$ is related to $\boldsymbol{u}^* - \partial X/\partial t$ in precisely the same way as the two magnetic field vectors above: though the concept of a frozen velocity is not quite so clear. Finally since $\boldsymbol{b}^*$, $\boldsymbol{u}^*$ and $\partial X/\partial t$ have zero divergence the nature of the transformation procedure implies that $\boldsymbol{b}$ and $\boldsymbol{u}$ also have zero divergence. More direct mathematical reasons will be given in the next section.

At this stage it is convenient to summarize the objectives and layout of the paper. Attention is restricted (with the exception § 5) to the magnetic induction equation which in dimensionless form is

$$\partial \boldsymbol{b}^*/\partial t = \nabla_X \times (\boldsymbol{u}^* \times \boldsymbol{b}^*) + R^{-1}\nabla_X^2 \boldsymbol{b}^*, \tag{1.35}$$

where the suffix X indicates that the gradient operator acts at the point X and both the magnetic field $\boldsymbol{b}^*(X,t)$ and the fluid velocity $\boldsymbol{u}^*(X,t)$ have zero divergence:

$$\nabla_X\cdot\boldsymbol{u}^* = \nabla_X\cdot\boldsymbol{b}^* = 0. \tag{1.36}$$

It is supposed that the magnetic Reynolds R is large:

$$R = \sigma\mu U^* L \gg 1, \tag{1.37}$$

where L is the radius of the spherical cavity and U^* is a typical fluid velocity. It should be noted that, unlike many of the turbulent theories, L is the only length scale of importance characterizing the problem. The velocity $\boldsymbol{u}^*$ is given and the problem is completely posed when suitable boundary conditions are applied. In § 2 the magnetic induction equation (1.35) is transformed with the help of (1.20) to (1.34) into an equation for $\boldsymbol{b}(x,t)$. The equation is separated into its mean and fluctuating parts. Part of the averaged diffusion term can be decomposed into a new diffusion term and an α-effect. One important component of the matrix associated with the α-effect is a measure of the helicity (see (2.52)). In § 3 the production of azimuthal magnetic field by shearing the meridional magnetic field is neglected so that the character of the α-effect may be considered in detail. In § 4 shearing of the magnetic field is considered in conjunction with the α-effect. In this case it is found that the $\langle\boldsymbol{\omega}\rangle\times\langle\boldsymbol{j}\rangle$-effect (see (1.9)) may also be significant. When the displacement of the loops is small (specifically $X(x,t) - x = O(R^{-\frac{1}{2}})$) an exact correspondence is made with the earlier work of Branginskiĭ (1964*a*). In § 5 the equation of motion is considered briefly in order to study the magnetic induction equation based on both an order 1 and an order R time scale. As mentioned earlier in the section, the fast (order 1) time scale is associated with the periods of a class of hydromagnetic waves that may be excited in the Earth's core while the slow (order R) time scale is associated with the free decay time. Dynamo equations relevant to the model are extracted from the equations derived in § 4. These equations correspond

† This representation for a 'frozen' vector function is well known (see, for example, Serrin 1959, §17).

to those obtained by Braginskiĭ (1964*b*) but have more general application. In § 6 consideration is given to the best choice of the transformation $X = X(x, t)$ which is far from being uniquely defined. Moreover, the large magnetic Reynolds number expansion may be invalid in some regions of the flow where the velocity is small or spatial derivatives are large. The first of these difficulties may never occur if the velocity is large everywhere including the neighbourhood of the boundary. This case is considered together with the boundary conditions at the fluid/solid interface and attention is given to the associated boundary layer. Finally a few concluding remarks are made in § 7.

2. The basic equations

(*a*) *Some preliminary identities*

Before considering the magnetic induction equation in detail some algebraic structure must be developed. Though equations (1.35), (1.36) are naturally functions of X, t they are regarded as functions of x, t, where $X = X(x, t)$. Therefore it is necessary to relate quantities like the gradient of a scalar $\psi(X, t)$ at $X(x, t)$, namely $\nabla_X \psi(X, t)$, to the gradient of ψ at x (regarded as a function of x), namely $\nabla\psi[X(x, t), t]$. Following the notation (1.22) the ith component of $\nabla\psi$ in cylindrical polar coordinates is

$$[\nabla\psi]_i = \partial_i \psi. \tag{2.1}$$

It should be noted that the suffix X is only inserted after the gradient operator when the operator is applied at X: the suffix x is omitted when applied at x. Moreover, it is clear that

$$\nabla\psi = (\nabla X)\cdot\nabla_X \psi(X, t),\dagger \tag{2.2}$$

and using the definition (1.26) the ith component of $\nabla\psi$ is

$$\partial_i \psi = a_{ji}(X)\,[\nabla_X \psi(X, t)]_j. \tag{2.3}$$

After multiplying (2.3) by the inverse of the matrix $a_{ji}(X)$, namely $T_{ki}(X)$, the identity

$$[\nabla_X \psi(X, t)]_i = T_{ij}(X)\,\partial_j \psi \tag{2.4}$$

follows immediately. It can also be shown for the vector functions $F(X, t)$ and $G(X, t)$ that

$$\nabla_X \cdot F = T_{ij}(X)\,a_{ij}(F), \tag{2.5}$$

$$[\nabla_X \times F]_i = \epsilon_{ijk}\,T_{jl}(X)\,a_{kl}(F), \tag{2.6}$$

$$[F\cdot\nabla_X G]_i = F_l\,T_{lk}(X)\,a_{ik}(G). \tag{2.7}$$

Finally it is clear that

$$\left(\frac{\partial F}{\partial t}\right)_{x=\text{const}} = \left(\frac{\partial F}{\partial t}\right)_{X=\text{const}} + \frac{\partial X}{\partial t}\cdot\nabla_X F,$$

and hence, by the use of (2.7), the ith component of $\partial F/\partial t$ (keeping X fixed) is

$$\left[\left(\frac{\partial F}{\partial t}\right)_{X=\text{const}}\right]_i = \left[\left(\frac{\partial F}{\partial t}\right)_{x=\text{const}}\right]_i - \frac{\partial X_j}{\partial t}\,T_{jk}(X)\,a_{ik}(F). \tag{2.8}$$

† Here ∇X is the gradient of a vector. In a rectangular Cartesian coordinate system the ith component of $\nabla\psi$ is

$$\frac{\partial\psi}{\partial x_i} = \frac{\partial X_j}{\partial x_i}\frac{\partial\psi}{\partial X_j}.$$

This identity is recovered from (2.3) if the terms introduced by curvature are neglected.

Some identities involving a_{ij} and T_{ij} are now introduced. From (1.27), (1.28) and (1.31) the algebraic identities

$$a_{ij}(\boldsymbol{X}) = (1/2!)\,\epsilon_{ilm}\,\epsilon_{jpq}\,T_{lp}(\boldsymbol{X})\,T_{mq}(\boldsymbol{X}), \tag{2.9}$$

and

$$\epsilon_{mjk}\,T_{mi}(\boldsymbol{X}) = \epsilon_{ipq}\,a_{jp}(\boldsymbol{X})\,a_{kq}(\boldsymbol{X}), \tag{2.10a}$$

$$\epsilon_{mjk}\,T_{im}(\boldsymbol{X}) = \epsilon_{ilm}\,a_{lj}(\boldsymbol{X})\,a_{mk}(\boldsymbol{X}) \tag{2.10b}$$

can be obtained. Note the interchangeability of a_{ij} and T_{ij} in the identity (2.9) (and consequently in (2.10)) depends on $J(\boldsymbol{X}) = 1$. A fundamental identity concerning the derivatives of a_{ij} is obtained by direct differentiation of $J(\boldsymbol{F})$, namely

$$\partial_m[J(\boldsymbol{F})] = T_{ij}(\boldsymbol{F})\,\partial_m[a_{ij}(\boldsymbol{F})]. \tag{2.11}$$

With the help of (1.27) and (1.32) it follows that

$$T_{ij}(\boldsymbol{X})\,\partial_m[a_{ij}(\boldsymbol{X})] = \partial_m[T_{ij}(\boldsymbol{X})]\,a_{ij}(\boldsymbol{X}) = 0. \tag{2.12}$$

Noting the elementary identity

$$\partial_i\partial_j - \partial_j\partial_i = -\epsilon_{3ij}(1/\rho)\,\partial_2, \tag{2.13}$$

it is readily established from the previous statements that

$$\partial_j[a_{ik}(\boldsymbol{F})] - \partial_k[a_{ij}(\boldsymbol{F})] = -\epsilon_{3jk}(1/\rho)\,a_{i2}(\boldsymbol{F}) + (1/\rho)\,\epsilon_{3ni}[\delta_{k2}a_{nj}(\boldsymbol{F}) - \delta_{j2}a_{nk}(\boldsymbol{F})], \tag{2.14a}$$

$$T_{ij}(\boldsymbol{X})\,\partial_j[a_{im}(\boldsymbol{X})] = -\partial_j[T_{ij}(\boldsymbol{X})]\,a_{im}(\boldsymbol{X}) = (1/\rho)\,\delta_{m1} - (1/\rho)\,\epsilon_{3ki}\,T_{i2}(\boldsymbol{X})\,a_{km}(\boldsymbol{X}), \tag{2.14b}$$

$$\partial_j[T_{ij}(\boldsymbol{X})] = -(1/\rho)\,\{T_{i1}(\boldsymbol{X}) + \epsilon_{3ni}\,T_{n2}(\boldsymbol{X})\}, \tag{2.14c}$$

$$\epsilon_{ipq}\,\partial_p[a_{jq}(\boldsymbol{X})] = -(1/\rho)\,\delta_{i3}\,a_{j2}(\boldsymbol{X}) + (1/\rho)\,\epsilon_{ip2}\,\epsilon_{3qj}\,a_{qp}(\boldsymbol{X}). \tag{2.14d}$$

Evidently all the terms on the right-hand side of (2.14) are curvature terms, i.e. in a rectangular Cartesian coordinate system none of these terms appear.

In the previous section it was argued on the basis of the transformation property of $\boldsymbol{b}(\boldsymbol{x},t)$ into $\boldsymbol{b}^*(\boldsymbol{X},t)$ that $\nabla\cdot\boldsymbol{b}$ is zero. Now a direct argument is given. From the definition (1.26) and the identities (2.5) and (2.14c) it follows that

$$\nabla_{\boldsymbol{X}}\cdot\boldsymbol{F} = (1/\rho)\,\partial_i[\rho T_{mi}(\boldsymbol{X})\,F_m]. \tag{2.15}$$

Hence from (1.33), noting $\nabla_{\boldsymbol{X}}\cdot\boldsymbol{b}^* = 0$, (2.15) leads to the identity

$$\nabla\cdot\boldsymbol{b}(\boldsymbol{x},t) = 0. \tag{2.16}$$

Similar arguments show that $\boldsymbol{u}(\boldsymbol{x},t)$, defined by (1.34) has zero divergence:

$$\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t) = 0, \tag{2.17}$$

since both $\nabla_{\boldsymbol{X}}\cdot\boldsymbol{u}^*$ and $\nabla_{\boldsymbol{X}}\cdot(\partial\boldsymbol{X}/\partial t)$ are zero.

Finally it can be shown, making particular use of (2.10) and (2.14d), that

$$\epsilon_{ijk}\,T_{im}(\boldsymbol{X})\,T_{jl}(\boldsymbol{X})\,a_{kl}(\boldsymbol{F}) = \epsilon_{mls}\,\partial_l[a_{ks}(\boldsymbol{X})\,F_k] + (1/\rho)\,\delta_{3m}\,a_{k2}(\boldsymbol{X})\,F_k. \tag{2.18}$$

Hence by (2.6) it follows that

$$T_{mi}(\boldsymbol{X})\,[\nabla_{\boldsymbol{X}}\times\boldsymbol{F}]_m = [\nabla\times\boldsymbol{f}]_i, \tag{2.19}$$

where

$$f_i(\boldsymbol{x},t) = a_{ki}(\boldsymbol{X})\,F_k(\boldsymbol{X},t). \tag{2.20}$$

(b) The transformation of the magnetic induction equation

The analysis of the magnetic induction equation is divided into two parts: (I) the advection term

$$(\text{advection})^* (\boldsymbol{X}, t) = (\partial \boldsymbol{b}^*/\partial t) - \nabla_{\boldsymbol{X}} \times (\boldsymbol{u}^* \times \boldsymbol{b}^*) \qquad (2.21a)$$

is considered, and (II) the diffusion term

$$(\text{diffusion})^* (\boldsymbol{X}, t) = \nabla^2_{\boldsymbol{X}} \boldsymbol{b}^* \qquad (2.21b)$$

is considered. In view of the results established between (2.15) and (2.20) it is natural to introduce the related advection and diffusion vectors in a similar manner to the related magnetic field vector $\boldsymbol{b}(\boldsymbol{x}, t)$ with, of course, similar interpretations. Thus from (2.21) the related vectors

$$(\text{advection})_i (\boldsymbol{x}, t) = T_{ji}(\boldsymbol{X}) \, [\partial \boldsymbol{b}^*/\partial t - \nabla_{\boldsymbol{X}} \times (\boldsymbol{u}^* \times \boldsymbol{b}^*)]_j \qquad (2.22a)$$

$$(\text{diffusion})_i (\boldsymbol{x}, t) = T_{ji}(\boldsymbol{X}) \, [\nabla^2_{\boldsymbol{X}} \boldsymbol{b}^*]_j, \qquad (2.22b)$$

are introduced.

The simplification of (2.22*a*) is now straightforward. From the definitions (1.33) and (1.34) it follows that

$$a_{ji}(\boldsymbol{X}) \, [(\boldsymbol{u}^* - (\partial \boldsymbol{X}/\partial t)) \times \boldsymbol{b}^*]_j = (\boldsymbol{u} \times \boldsymbol{b})_i, \qquad (2.23)$$

and hence together with (2.19), (2.20) leads to

$$T_{ji}(\boldsymbol{X}) \, [\nabla_{\boldsymbol{X}} \times \{(\boldsymbol{u}^* - \partial \boldsymbol{X}/\partial t) \times \boldsymbol{b}^*\}]_j = [\nabla \times (\boldsymbol{u} \times \boldsymbol{b})]_i. \qquad (2.24)$$

Similar arguments making use of (2.8) give

$$T_{ji}(\boldsymbol{X}) \left[\frac{\partial \boldsymbol{b}^*}{\partial t} - \nabla_{\boldsymbol{X}} \times \left(\frac{\partial \boldsymbol{X}}{\partial t} \times \boldsymbol{b}^* \right) \right]_j = \left[\frac{\partial \boldsymbol{b}}{\partial t} \right]_i. \qquad (2.25)$$

Hence combining (2.24) and (2.25) leads to

$$(\text{advection}) \, (\boldsymbol{x}, t) = \partial \boldsymbol{b}/\partial t - \nabla \times (\boldsymbol{u} \times \boldsymbol{b}). \qquad (2.26)$$

This simple result could perhaps have been expected owing to the physical nature of the advection terms and owing to the original definitions of $\boldsymbol{u}$ and $\boldsymbol{b}$. Thus after considerable algebra the advection terms look neither better nor worse than before! Clearly an advantage has been gained as only simple $\boldsymbol{u}$ need be considered later.

Though the advection terms have provided no problem, the expression for the diffusion term is more complicated. Making use of (2.19) and (2.20), we can express the related diffusion term (2.22*b*) in the form

$$(\text{diffusion}) \, (\boldsymbol{x}, t) = -\nabla \times \mathscr{E}, \qquad (2.27)$$

where
$$\mathscr{E}_i(\boldsymbol{x}, t) = a_{ji}(\boldsymbol{X}) \, [\nabla_{\boldsymbol{X}} \times \boldsymbol{b}^*]_j. \qquad (2.28)$$

After lengthy but straightforward algebraic manipulations (outlined in appendix A) the following expression for $\mathscr{E}$ is obtained:

$$\mathscr{E}_i(\boldsymbol{x}, t) = \epsilon_{ipq} \alpha_{kp}(\boldsymbol{X}) \, \partial_k b_q - \mu_{ij}(\boldsymbol{X}) \, b_j, \qquad (2.29)$$

where $\alpha_{ij}(\boldsymbol{X})$ and $\mu_{ij}(\boldsymbol{X})$ are defined by

$$\alpha_{ij}(\boldsymbol{X}) = T_{pi}(\boldsymbol{X}) \, T_{pj}(\boldsymbol{X}), \qquad (2.30a)$$

and

$$\mu_{ij}(\boldsymbol{X}) = \epsilon_{ikl} T_{pk}(\boldsymbol{X}) \, \partial_j [T_{pl}(\boldsymbol{X})] - (1/\rho) \, \{\epsilon_{ikl} \delta_{j2} \epsilon_{3pq} + \epsilon_{ik2} \epsilon_{jl3} \delta_{pq}\} \, T_{pk}(\boldsymbol{X}) \, T_{ql}(\boldsymbol{X}). \qquad (2.30b)$$

After further manipulation the matrices $\alpha_{ij}(X)$ and $\mu_{ij}(X)$ may be represented in the alternative form

$$\alpha_{ij}(X) = l^{(i)}\cdot l^{(j)}, \tag{2.31a}$$

$$\mu_{ij}(X) = -(\partial_i X)\cdot[\nabla_X \times (\partial_j X)], \tag{2.31b}$$

where
$$l^{(i)}(X) = \tfrac{1}{2}\epsilon_{ipq}(\partial_p X)\times(\partial_q X). \tag{2.31c}$$

Moreover, since $J(X) = 1$, the vectors $l^{(i)}$ have the property

$$l^{(1)}\cdot[l^{(2)}\times l^{(3)}] = 1. \tag{2.32}$$

Clearly the representations (2.31) are more readily interpreted than (2.30). Finally, it is shown in appendix A that $\alpha_{ij}(X)$ and $\mu_{ij}(X)$ are related by the identity

$$(1/\rho)\,\partial_j[\rho\alpha_{ij}(X)] = \epsilon_{ipq}\mu_{pq}(X) = \nabla_X\cdot l^{(i)}. \tag{2.33}$$

It follows that the full magnetic induction equation becomes

$$\partial b/\partial t = \nabla\times(u\times b) - R^{-1}\nabla\times\mathscr{E}, \tag{2.34}$$

where $\mathscr{E}$ is defined by (2.29).

So far no assumptions have been made about the velocity field $u^*(X, t)$. However, attention is subsequently restricted to velocity fields with the property that a coordinate transformation $X = X(x, t)$ can be found such that

(i) $a_{ij}(X), \partial_k[a_{ij}(X)]$ and $\partial_m\partial_k[a_{ij}(X)]$ are order 1,

so that approximations based on $R \gg 1$ are valid and

(ii) the flow is predominantly azimuthal, specifically

$$u(x, t) = U(\rho, z, t)\,i_\phi + O(R^{-1}).$$

In this paper azimuthal and meridional refer to the related velocity field $u(x, t)$. Soward (1971*b*) has shown that the flows considered by Braginskiĭ (1964*a*, *b*) are two such examples. In these two cases the displacement vector $X(x, t) - x$ required to satisfy conditions (i) and (ii) is order $R^{-\frac{1}{2}}$. In general, condition (ii) implies that the instantaneous streamlines are almost coincident with the loops $C(X)$ provided $\partial X/\partial t$ is small. Moreover, in contrast with Braginskiĭ's model the velocity field $u^*(X, t)$ need not be nearly axisymmetric. Instead it is the *related* velocity field $u(x, t)$ which is required to be predominately axisymmetric and azimuthal. For a velocity field $u^*(X, t)$ which satisfies conditions (i) and (ii), the coordinate transformation $X(x, t)$ is not uniquely defined. This problem and the related difficulties associated with the boundary conditions are considered in §6. It should be noted that, though the Lagrangian description of the flow (characterized by $u(x, t) = 0$) is attractive, condition (i) is unlikely to be satisfied as well. In general the large velocities will wrap up the coordinate system $X(x, t)$ so that the quantities listed in (i) will become very large. Indeed on the order R time scale (the free decay time) with which the paper is principally concerned it is to be expected that these quantities would become order R in that time. This last remark emphasizes the restriction that condition (i) imposes on the velocity field $u^*(X, t)$ as (i) can always be satisfied for a sufficiently short interval of time.

The related velocity and magnetic field vectors u and b are separated into symmetric and asymmetric parts by defining

$$u(x, t) = U(\rho, z, t) + R^{-1}u'(x, t), \tag{2.35a}$$

$$b(x, t) = B(\rho, z, t) + R^{-1}b'(x, t), \tag{2.35b}$$

where primed quantities and the ϕ-average are defined by (1.13), so that $\boldsymbol{U} = \langle \boldsymbol{u} \rangle$ and $\boldsymbol{B} = \langle \boldsymbol{b} \rangle$. Further $\boldsymbol{U}$ and $\boldsymbol{B}$ are separated into their azimuthal and meridional components

$$\boldsymbol{U}(\rho, z, t) = U(\rho, z, t)\, \boldsymbol{i}_\phi + R^{-1} \boldsymbol{u}_{\mathrm{M}}(\rho, z, t), \tag{2.36a}$$

$$\boldsymbol{B}(\rho, z, t) = B(\rho, z, t)\, \boldsymbol{i}_\phi + \nabla \times [\chi(\rho, z, t)\, \boldsymbol{i}_\phi], \tag{2.36b}$$

$$= \left(-\frac{\partial \chi}{\partial z},\ B,\ \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\chi) \right).$$

The ordering for $\boldsymbol{u}(\boldsymbol{x}, t)$ as defined by (2.35a), (2.36a) is a consequence of assumption (ii), while the choice of ordering for $\boldsymbol{b}(\boldsymbol{x}, t)$ will become apparent later. Taking the ϕ-average of (2.34) gives

$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) - R^{-1} \nabla \times \mathscr{E}_0 + R^{-2} \nabla \times \{\langle \boldsymbol{u}' \times \boldsymbol{b}' \rangle - \mathscr{E}_1\}, \tag{2.37}$$

and subtracting this identity from (2.34) gives

$$\partial \boldsymbol{b}'/\partial t = \nabla \times [\boldsymbol{u}' \times \boldsymbol{B} + \boldsymbol{U} \times \boldsymbol{b}' + R^{-1}(\boldsymbol{u}' \times \boldsymbol{b}' - \langle \boldsymbol{u}' \times \boldsymbol{b}' \rangle)] - \nabla \times \mathscr{E}'. \tag{2.38}$$

The vectors $\mathscr{E}_0$, $\mathscr{E}_1$ and $\mathscr{E}'$ are defined by

$$[\mathscr{E}_0]_i = \epsilon_{ipq} \langle \alpha_{kp}(\boldsymbol{X}) \rangle\, \partial_k B_q - \langle \mu_{ij}(\boldsymbol{X}) \rangle\, B_j, \tag{2.39a}$$

$$[\mathscr{E}_1]_i = \epsilon_{ipq} \langle \alpha'_{kp}(\boldsymbol{X})\, \partial_k b'_q \rangle - \langle \mu'_{ij}(\boldsymbol{X})\, b'_j \rangle, \tag{2.39b}$$

and
$$\mathscr{E}' = \mathscr{E} - \mathscr{E}_0 - R^{-1} \mathscr{E}_1, \tag{2.39c}$$

where the prime operator extracts the fluctuating part, i.e.

$$\psi'(\boldsymbol{x}, t) = \psi(\boldsymbol{x}, t) - \langle \psi(\boldsymbol{x}, t) \rangle. \tag{2.40}$$

The main objective is to solve (2.37), (2.38) for the mean related magnetic field vector $\boldsymbol{B}$ by an iterative procedure: the first approximation neglects the last terms in (2.37) which result from the ϕ-average of fluctuating quantities. Indeed Braginskiĭ's (1964a, b) equations can be extracted without considering these terms at all! Clearly $\boldsymbol{B}(\rho, z, t)$ is not related to the magnetic field in a simple way. However, the part of the magnetic field corresponding to $\boldsymbol{B}$ can easily be obtained from (1.33) and, if the part corresponding to $\boldsymbol{b}'(\boldsymbol{x}, t)$, which may be obtained from the iteration procedure, is also significant, the total field $\boldsymbol{b}^*(\boldsymbol{x}, t)$ is determined from $\boldsymbol{B} + R^{-1}\boldsymbol{b}'$ in the usual way.

The character of $\nabla \times \mathscr{E}_0$ appearing in (2.37) is now considered in detail. In appendix B is it shown that

$$-[\mathscr{E}_0]_\phi = \partial_i[(1/\rho)\, A_{ij}\, \partial_j(\rho\chi)] + \Gamma_{2j} B_j, \tag{2.41}$$

and
$$-[\nabla \times \mathscr{E}_0]_\phi = \partial_i[(1/\rho)\, A_{ij}\, \partial_j(\rho B)] + \epsilon_{2ij}\, \partial_i(\Gamma_{jk} B_k), \tag{2.42}$$

where A_{ij} is the symmetric matrix

$$A_{ij}(\rho, z, t) = \langle \alpha_{ij}(\boldsymbol{X}) \rangle, \tag{2.43}$$

and Γ_{ij} is the symmetric matrix

$$\Gamma_{ij}(\rho, z, t) = \langle \gamma_{ij}(\boldsymbol{X}) \rangle, \tag{2.44}$$

$$\gamma_{ij}(\boldsymbol{X}) = \tfrac{1}{2}\epsilon_{2pq}\{\epsilon_{2jn}\epsilon_{pim} + \epsilon_{2in}\epsilon_{pjm}\}\, \rho\, \partial_q[(1/\rho)\, \alpha_{mn}(\boldsymbol{X})] + \tfrac{1}{2}\{\mu_{ij}(\boldsymbol{X}) + \mu_{ji}(\boldsymbol{X})\}. \tag{2.45}$$

By considering the quadratic form

$$y_i A_{ij} y_j = \langle (T_{pi}(\boldsymbol{X})\, y_i)\, (T_{pj}(\boldsymbol{X})\, y_j) \rangle > 0,$$

where $\boldsymbol{y}$ is a non-zero constant vector, it is clear that the matrix A_{ij} is positive definite. With the use of (2.41), (2.42) the meridional and azimuthal components of (2.37) give respectively

$$R\frac{\partial \chi}{\partial t}+\frac{1}{\rho}\boldsymbol{u}_{\mathrm{M}}\cdot\nabla(\rho\chi) = \Gamma_{2j}B_j+\partial_i\left[\frac{1}{\rho}A_{ij}\partial_j(\rho\chi)\right]+R^{-1}\{\langle\boldsymbol{u}'\times\boldsymbol{b}'\rangle-\mathscr{E}_1\}_{\phi}, \tag{2.46}$$

and
$$R\frac{\partial B}{\partial t}+\rho\boldsymbol{u}_{\mathrm{M}}\cdot\nabla\frac{B}{\rho} = R\left(\nabla\frac{U}{\rho}\times\nabla\rho\chi\right)_{\phi}+\epsilon_{2ij}\partial_i(\Gamma_{jk}B_k)+\partial_i\left[\frac{1}{\rho}A_{ij}\partial_j(\rho B)\right] +R^{-1}[\nabla\times\{\langle\boldsymbol{u}'\times\boldsymbol{b}'\rangle-\mathscr{E}_1\}]_{\phi}. \tag{2.47}$$

The terms on the left hand side of (2.46), (2.47) are well known and represent the advection of $\rho\chi$ and B/ρ by the meridional velocity $\boldsymbol{u}_{\mathrm{M}}$. The term $\nabla(U/\rho)\times\nabla\rho\chi$ represents the mechanism for creating azimuthal magnetic field by shearing the meridional magnetic field. However, the terms involving A_{ij} and Γ_{ij} are new.

Braginskiĭ (1964*a*) provided a simple proof that axisymmetric dynamos are impossible. For the particular case where the loops $C(\boldsymbol{X})$ are the circles $C(\boldsymbol{x})$, A_{ij} becomes δ_{ij} and Γ_{ij} vanishes. By hypothesis in an axisymmetric model the mean of fluctuating quantities in (2.46), (2.47) does not appear. Multiplying (2.46) by $\rho^2\chi$, (2.47) by B/ρ^2 and integrating throughout the cavity V_1 (the exterior V_2 is assumed to be a vacuum), Braginskiĭ showed that the time derivatives of

$$\int_{V_1+V_2}(\chi^2/\rho^2)\,\mathrm{d}v \quad\text{and}\quad \int_{V_1}\rho^2B^2\,\mathrm{d}v$$

are negative. The proof depends on forming strictly negative quantities from the diffusion terms. By a similar argument it is now shown that in the general case the terms containing A_{ij} explicitly in (2.46), (2.47) can be regarded as strictly diffusive in the sense that they cannot contribute to the increase of

$$\int_{V_1+V_2}\chi^2\,\mathrm{d}v \quad\text{and}\quad \int_{V_1}B^2\,\mathrm{d}v,$$

provided $\boldsymbol{X}(\boldsymbol{x},t) = \boldsymbol{x}$ on and near the boundary. The last restriction is to insure that $A_{ij} = \delta_{ij}$ on the boundary. In order to determine the rate of change

$$\int_{V_1}\chi^2\,\mathrm{d}v \quad\text{and}\quad \int_{V_1}B^2\,\mathrm{d}v,$$

(2.46), (2.47) are multiplied by χ and B respectively and integrated throughout V_1. The contribution from the A_{ij} terms can be expressed in the form

$$\int_S A_{ij}\frac{\chi}{\rho}\partial_j(\rho\chi)\,\mathrm{d}S_i-\int_{V_1}\frac{1}{\rho^2}A_{ij}\partial_i(\rho\chi)\,\partial_j(\rho\chi)\,\mathrm{d}v, \tag{2.48a}$$

and
$$\int_S A_{ij}\frac{B}{\rho}\partial_j(\rho B)\,\mathrm{d}S_i-\int_{V_1}\frac{1}{\rho^2}A_{ij}\,\partial_i(\rho B)\,\partial_j(\rho B)\,\mathrm{d}v, \tag{2.48b}$$

where S is the surface bounding the fluid. Since $A_{ij} = \delta_{ij}$ and $\boldsymbol{X} = \boldsymbol{x}$ on the boundary the surface integral involving χ can be transformed into an integral throughout the exterior region. Also, since $B = 0$ on the boundary, the surface integral involving B vanishes. Hence from (2.48) the contribution to the rate of change of

$$\int_{V_1+V_2}\chi^2\,\mathrm{d}v \quad\text{and}\quad \int_{V_1}B^2\,\mathrm{d}v$$

is respectively
$$-\int_{V_1+V_2}\frac{1}{\rho^2}A_{ij}\,\partial_i(\rho\chi)\,\partial_j(\rho\chi)\,\mathrm{d}v \leqslant 0. \tag{2.49a}$$
and
$$-\int_{V_1}\frac{1}{\rho^2}A_{ij}\,\partial_i(\rho B)\,\partial_j(\rho B)\,\mathrm{d}v \leqslant 0, \tag{2.49b}$$
where $\boldsymbol{X}(\boldsymbol{x},t)=\boldsymbol{x}$ in the exterior region. The integrals are negative since the matrix A_{ij} is positive definite. Further A_{ij} appears in (2.46), (2.47) as an anisotropic diffusivity (dependent on position) characterizing the diffusion of χ and B.

Evidently the terms involving Γ_{ij} correspond to the α-effect (see (1.10)) introduced in § 1. The correspondence is made explicit by defining
$$\nabla\times\mathscr{E}_0^{(\Gamma)}=\nabla\times\boldsymbol{J}, \tag{2.50}$$
where $\mathscr{E}_0^{(\Gamma)}$ is the part of $\mathscr{E}_0$ associated with the matrix Γ_{ij}, and
$$J_i(\boldsymbol{x},t)=-\Gamma_{ij}B_j. \tag{2.51}$$
The vector $\boldsymbol{J}$ may be regarded as an electric current which is linearly related to the related magnetic field vector $\boldsymbol{B}$. (Note that in general $\nabla\cdot\boldsymbol{J}\neq 0$, and $\boldsymbol{J}\neq\mathscr{E}_0^{(\Gamma)}$.) Though the matrix corresponding to Γ_{ij} obtained for homogeneous turbulence by Moffatt (1970a) is symmetric and though a similar remark is true for the spatially periodic dynamos of Childress (1969, 1970) (with one proviso, see § 1), it seems remarkable to the author that Γ_{ij} appearing in (2.51) should be symmetric, especially in view of curvature effects. Most of the components of Γ_{ij} do not have a simple interpretation. However, Γ_{22} is strikingly simple. Since $\gamma_{22}(\boldsymbol{X})=\mu_{22}(\boldsymbol{X})$, it follows from (2.31b) that
$$-\Gamma_{22}(\rho,z,t)=\frac{1}{2\pi\rho}\oint_{C(X)}\mathrm{d}\boldsymbol{X}\cdot\left[\nabla_{\boldsymbol{X}}\times\frac{1}{\rho}\frac{\partial\boldsymbol{X}}{\partial\phi}\right]. \tag{2.52}$$
Thus Γ_{22} is a measure of the helicity of the contour $C(\boldsymbol{X})$ and as in the case of turbulent dynamos the helicity plays a central role in the present theory. It follows from (2.51) that corresponding to Γ_{22} there is an electric current
$$J_2\boldsymbol{i}_\phi=-\Gamma_{22}B\boldsymbol{i}_\phi. \tag{2.53}$$
This current can be given a physical interpretation. The magnetic field corresponding to $\boldsymbol{b}(\boldsymbol{x},t)=B\boldsymbol{i}_\phi$ represents magnetic field alined to the loops $C(\boldsymbol{X})$. In fact it is the magnetic field that results after the axisymmetric magnetic field is moved instantaneously from the loops $C(\boldsymbol{x})$ to $C(\boldsymbol{X})$. Clearly if the loops have a preferred twist as measured by Γ_{22} there is, associated with the diffusion of the magnetic field, an electric current in the azimuthal direction equal to $-\Gamma_{22}B$. This electric current may be significant in regenerating meridional magnetic field. As mentioned in § 1, this mechanism is fundamental in turbulent dynamo models, especially $\alpha\omega$-dynamos. Thus the compact representation for Γ_{22} given by (2.52), measuring the twist of the loops $C(\boldsymbol{X})$, is truly significant. Finally it is shown in appendix C that $R\Gamma_{22}$ is the same as the Γ introduced by Braginskiĭ (1964a, b) for the special case where $\boldsymbol{X}(\boldsymbol{x},t)-\boldsymbol{x}$ is order $R^{-\frac{1}{2}}$.

It should be emphasized at this stage that the averaged equations (2.46), (2.47) are consistency conditions (in the sense of § 1) resulting from integration around the loops $C(\boldsymbol{X})$. Though the loops are not necessarily streamlines the component of velocity normal to the loops is order R^{-1} provided $\partial\boldsymbol{X}/\partial t$ is small. Thus advective effects resulting from this component of velocity are of the same order of magnitude as the diffusive effects. Consequently, instead of obtaining almost trivial equations such as (1.19) (incidently (1.19) is a special case of (2.46)), consistency conditions lead to equations almost as elaborate as the initial equations (1.35), (1.36). However, the averaged equations are dependent on one less space coordinate. This simplicity leads to practical advantages and makes the mechanisms involved conceptually easier to understand.

3. α^2-Dynamos

In this section the possibility of dynamo models sustained by the electric current $\boldsymbol{J}(\boldsymbol{x}, t)$ given by (2.51) is considered. In order to isolate this mechanism it will be supposed that

$$\boldsymbol{u}' = 0, \quad \nabla(U/\rho) = 0, \tag{3.1}$$

and that quantities vary on a (slow) time scale order R. There is no reason to suppose that solutions do not exist with $R^{-1}\boldsymbol{b}'(\boldsymbol{x}, t)$, as defined by (2.35$b$), order 1. However, if this is the case (2.38) gives

$$\boldsymbol{b}'(\boldsymbol{x}, t) = \boldsymbol{b}'(\rho, \phi - (U/\rho)\, t, z), \tag{3.2}$$

as the first approximation of the fluctuating magnetic field. This possibility is not considered further: in any case it violates the assumption that quantities vary on a slow time scale. Thus the mean part of the related magnetic field (assuming the ordering given by (2.35), (2.36)) is determined from the equations

$$R\frac{\partial \chi}{\partial t} + \frac{1}{\rho}\boldsymbol{u}_{\mathrm{M}} \cdot \nabla(\rho\chi) = \Gamma_{2j} B_j + \partial_i \left[\frac{1}{\rho} A_{ij}\, \partial_j(\rho\chi)\right] + O(R^{-1}), \tag{3.3a}$$

$$R\frac{\partial B}{\partial t} + \rho\boldsymbol{u}_{\mathrm{M}} \cdot \nabla\frac{B}{\rho} = \epsilon_{2ij}\, \partial_i(\Gamma_{jk} B_k) + \partial_i \left[\frac{1}{\rho} A_{ij}\, \partial_j(\rho B)\right] + O(R^{-1}), \tag{3.3b}$$

while the order R^{-1} fluctuating field may be determined subsequently from (2.38). This model relies solely on the α-effect $J_i = -\Gamma_{ij} B_j$ for its maintenance and so is called an α^2-dynamo (see § 1).

A curious feature of (3.3) is that to lowest order the equations are independent of the magnitude of the velocity. However, they do depend on the direction of the velocity via the coefficients Γ_{ij} and A_{ij}. Thus dynamos of this type are insensitive to the value of the related velocity $U(\rho, z, t)\, \boldsymbol{i}_\phi$ provided the large magnetic Reynolds number approximation is justified. In spite of the fact that U/ρ constant describes a solid body rotation, it must be emphasized that this refers to the related velocity and not the real fluid motion. On the other hand, it does imply that no azimuthal magnetic field is created by shearing the meridional magnetic field. The generation mechanism may be given a somewhat similar interpretation to that given for the generation mechanism $\Gamma_{22} B$ in the previous section. It is evident from the definition of the related magnetic field that to lowest order the 'frozen' field approximation has been made. The magnetic field is distorted owing to the character of the loops $C(\boldsymbol{X})$: the subsequent diffusion of this field may be capable of regenerating itself (as measured by the α-effect $\Gamma_{ij} B_j$). It is conceptually difficult to envisage how an effect, which appears to rely principally on diffusion is capable of self-excitation. Of course the large velocities are essential to the mechanism since the field must be convected into a configuration which is capable of dynamo action. This idea is similar to Parker's (1955), where $\Gamma_{22} B$ is interpreted as twisting the azimuthal field and subsequently allowing it to diffuse.

In order to determine whether dynamos exist satisfying (3.3) an explicit form of the transformation $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t)$ should be taken. The coefficients Γ_{ij} and A_{ij} may then be determined. This procedure is very awkward. Instead it is assumed that the matrices Γ_{ij} and A_{ij} are independent of position and time. It is possible that no transformations exist (except the trivial one $\boldsymbol{X} = \boldsymbol{x}$) which satisfy these conditions. However, even if this is the case it is hoped that the simplified model will give insight into the mechanisms involved and lead to a qualitative interpretation of (3.3). These approximations are similar to those that have been made for turbulent dynamo models (see § 1), though the motivation is different.

Two further approximations are made. First the cylindrical geometry is approximated by plane geometry by setting

$$x = \rho - \rho_0, \quad y = \rho_0 \phi, \tag{3.4}$$

and taking the limit $\rho_0 \to \infty$. Secondly, the flow is supposed unbounded so that no boundary conditions need be applied. Instead solutions are sought periodic in the space coordinates x and z. Thus a continuous, rather than a discrete, spectrum is obtained. In view of the severe assumptions made in setting Γ_{ij} and A_{ij} constant, there is little point in making improvements to these last two approximations. Finally for simplicity it is supposed that

$$\boldsymbol{u}_{\mathrm{M}} = 0. \tag{3.5}$$

On the basis of the foregoing assumptions and approximations, (3.3) becomes

$$R\,\partial \boldsymbol{B}/\partial t = A_{ij}\,\partial_i\,\partial_j\,\boldsymbol{B} - \nabla \times \boldsymbol{J}, \tag{3.6}$$

where
$$J_i = -\Gamma_{ij} B_j \quad \text{and} \quad \nabla \cdot \boldsymbol{B} = 0. \tag{3.7}$$

A solution of (3.6), (3.7) is sought proportional to

$$\exp\,(nt/R - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}), \tag{3.8}$$

where
$$\boldsymbol{k} = (k_1, 0, k_3). \tag{3.9}$$

Direct substitution leads to the algebraic equations

$$\left.\begin{aligned}\{(n + k_p A_{pq} k_q)\,\delta_{ij} + \mathrm{i}\epsilon_{ilm} k_l \Gamma_{mj}\}\,B_j = D_{ij} B_j = 0,\\ k_i B_i = 0.\end{aligned}\right\} \tag{3.10}$$

This pair of equations has a non trivial solution provided

$$D^{\dagger}_{ij} k_j = 0, \tag{3.11}$$

where $D^{\dagger}_{ij}$ is the matrix of cofactors of D_{ij}, namely $D^{\dagger}_{ij} = \frac{1}{2}\epsilon_{ilm}\epsilon_{jpq} D_{lp} D_{mq}$. After direct substitution of D_{ij} given by (3.10), (3.11) becomes

$$\{(n + k_p A_{pq} k_q)^2 - k_p \Gamma^{\dagger}_{pq} k_q\}\,k_i = 0, \tag{3.12}$$

where $\Gamma^{\dagger}_{ij}$ is the matrix of cofactors of Γ_{ij}. It is important to note that the symmetry of Γ_{ij} has been used explicitly in the determination of (3.12). Provided that $\boldsymbol{k}$ is a non-zero vector, the dispersion relation

$$n = -k_i A_{ij} k_j \pm \{k_i \Gamma^{\dagger}_{ij} k_j\}^{\frac{1}{2}}, \tag{3.13}$$

follows immediately.

Dispersion relations of this type are not new (cf. Childress 1970, eqn (4.11); Moffatt 1970*a*, eqn (6.4)). However, two features distinguish it from earlier work. First, the diffusion coefficient A_{ij} is anisotropic – note that $k_i A_{ij} k_j > 0$. Secondly the vector $\boldsymbol{k}$ is restricted to lie in meridional planes (see (3.9)). Now by rotating and stretching the coordinates the matrix A_{ij} can be made isotropic (proportional to δ_{ij}) and $\Gamma^{\dagger}_{ij}$ can be diagonalized. However, in view of the second restriction little advantage is gained by making the transformation.

Clearly dynamo action ($\mathrm{Re}\,n \geqslant 0$) is only possible if one root of $(k_i \Gamma^{\dagger}_{ij} k_j)^{\frac{1}{2}}$ is strictly positive (real). Moreover, this quantity must be larger than $k_i A_{ij} k_j$. This is always possible if $|\boldsymbol{k}|$ is sufficiently small. An order of magnitude estimate indicates that, if the fluid is contained by boundaries (the dimensions of the cavity being order 1), $|\boldsymbol{k}|$ must be at least order 1. Hence for the particular problem considered here (unbounded), a necessary and sufficient condition for dynamo action is that one eigenvalue of the 2×2 matrix $\Gamma^{\dagger}_{ij}$ ($i \neq 2, j \neq 2$) is greater than zero. However, for a bounded fluid this condition is necessary but not sufficient. Thus for the problem of fluid bounded in a spherical cavity the displacement vector $\boldsymbol{X}(\boldsymbol{x}, t) - \boldsymbol{x}$ must be order 1. Otherwise, if $|\boldsymbol{X}(\boldsymbol{x}, t) - \boldsymbol{x}| \ll 1$, then

$$|A_{ij} - \delta_{ij}| \ll 1, \quad |\Gamma^{\dagger}_{ij}| \ll 1, \tag{3.14}$$

and in view of the above arguments dynamo action is unlikely.

The value of n takes on a different character depending on whether (*a*) $\Gamma^{\dagger}_{ij}$ is negative definite, (*b*) $\Gamma^{\dagger}_{ij}$ has one positive and one negative eigenvalue, (*c*) $\Gamma^{\dagger}_{ij}$ is positive definite ($i \neq 2, j \neq 2$). In case (*a*) $\{k_i \Gamma^{\dagger}_{ij} k_j\}^{\frac{1}{2}}$ is pure imaginary and the decay rate $\mathrm{Re}\,n = -k_i A_{ij} k_j$ is unaltered by the α-effect. In case (*c*) one mode always decays and one mode always grows if $|\boldsymbol{k}|$ is sufficiently small. Case (*b*) displays either the features of case (*a*) or case (*c*) depending on the direction of $\boldsymbol{k}$. The two eigenvalues $\lambda^{(i)}$ ($i = 1, 2$) may be determined explicitly in the form

$$2\lambda^{(1,2)} = -(\Gamma^2_{23} + \Gamma^2_{12}) + \Gamma_{22}(\Gamma_{11} + \Gamma_{33}) \pm \{(\Gamma^2_{23} + \Gamma^2_{12})^2 + \Gamma^2_{22}[(\Gamma_{33} - \Gamma_{11})^2 + 4\Gamma^2_{13}] + 2\Gamma_{22}[(\Gamma_{33} - \Gamma_{11})(\Gamma^2_{12} - \Gamma^2_{23}) - 4\Gamma_{23}\Gamma_{31}\Gamma_{12}]\}^{\frac{1}{2}}. \tag{3.15}$$

Evidently if $\Gamma_{22} = 0$, then
$$\lambda^{(1)} = 0, \quad \lambda^{(2)} = -(\Gamma^2_{23} + \Gamma^2_{12}), \tag{3.16}$$

so that dynamo action is impossible. Indeed the decay time is unaltered by the α-effect. This simple result emphasizes the important role played by the helicity Γ_{22} (see (2.52)) in maintaining the dynamo. Evidently when $\Gamma_{22} = 0$, (3.3*a*) indicates that there is no mechanism for creating meridional field from azimuthal field. It seems likely that the coupling of equations (3.3*a*) and (3.3*b*) is essential to the dynamo mechanism. The reason for this remark is made more apparent by writing (3.3*a*) in the form

$$R\frac{\partial \chi}{\partial t} + \frac{1}{\rho}\boldsymbol{u}^{(\pm)}_{\mathrm{M}} \cdot \nabla(\rho\chi) = \Gamma_{22} B + \partial_i \left[\frac{1}{\rho} A_{ij} \partial_j(\rho\chi)\right], \tag{3.17}$$

where the velocity $\boldsymbol{u}^{(\pm)}_{\mathrm{M}}$ is defined by

$$u^{(\pm)}_{\mathrm{M}i} = u_{\mathrm{M}i} \pm \epsilon_{2ij}\Gamma_{2j}. \tag{3.18}$$

In general $\nabla \cdot \boldsymbol{u}^{(\pm)}_{\mathrm{M}} \neq 0$, but provided that $\boldsymbol{X} = \boldsymbol{x}$ on and near the boundary S of the cavity, so that $\Gamma_{ij} \to 0$ on S, it follows that the velocity component of $\boldsymbol{u}^{(\pm)}_{\mathrm{M}}$ normal to the boundary vanishes also. Consequently $\boldsymbol{u}^{(\pm)}_{\mathrm{M}}$ describes a compressible flow contained in the cavity without sources or sinks. Thus in the absence of Γ_{22} it is likely that $\boldsymbol{u}^{(\pm)}_{\mathrm{M}}$ will redistribute the meridional magnetic field but not sustain it against ohmic decay.

For the case in which $\Gamma_{23} = \Gamma_{12} = 0$, the eigenvalues given by (3.15) are

$$\lambda^{(1,2)} = \tfrac{1}{2}\Gamma_{22}\{(\Gamma_{11} + \Gamma_{33}) \pm [(\Gamma_{11} - \Gamma_{33})^2 + 4\Gamma^2_{13}]^{\frac{1}{2}}\}. \tag{3.19}$$

Thus at least one eigenvalue is positive if either

$$\Gamma_{22}\Gamma_{11} > 0, \quad \Gamma_{22}\Gamma_{33} > 0, \quad \text{or} \quad \Gamma^2_{13} > \Gamma_{11}\Gamma_{33}. \tag{3.20}$$

For the general case a sufficient condition for a positive eigenvalue is that

$$\Gamma_{22}(\Gamma_{11} + \Gamma_{13}) > \Gamma^2_{23} + \Gamma^2_{12}. \tag{3.21}$$

This statement suggests that Γ_{23} and Γ_{12} may inhibit dynamo action (see also (4.22)).

The conclusions that have been drawn from this simple model should be interpreted loosely owing to the strong assumptions concerning Γ_{ij} and A_{ij}. However, the analysis does indicate the importance of various components of the matrix Γ_{ij} and clearly demonstrates the possibility of dynamo action.

Finally some symmetry properties of the model for the spherical cavity are considered. If it is supposed that

$$X_i(\rho, \phi, z, t) = \sigma(i)\, X_i(\rho, \phi, -z, t), \tag{3.22}$$

where

$$\sigma(1) = \sigma(2) = 1, \quad \sigma(3) = -1, \tag{3.23}$$

then it follows from the definitions of A_{ij} and Γ_{ij} that

$$A_{ij}(\rho, z, t) = \sigma(i)\, \sigma(j)\, A_{ij}(\rho, -z, t), \tag{3.24a}$$

and

$$\Gamma_{ij}(\rho, z, t) = -\sigma(i)\, \sigma(j)\, \Gamma_{ij}(\rho, -z, t). \tag{3.24b}$$

Hence a solution of (3.3) can be sought in the form

$$\left.\begin{aligned} B(\rho, z, t) &= -B(\rho, -z, t), \\ \chi(\rho, z, t) &= \chi(\rho, -z, t). \end{aligned}\right\} \tag{3.25}$$

Moreover, formally defining the parameters $\lambda^{(1,2)}$ by (3.15) it is clear that

$$\lambda^{(1,2)}(\rho, z, t) = \lambda^{(1,2)}(\rho, -z, t). \tag{3.26}$$

Hence, if $\lambda^{(i)} > 0$ ($i = 1$ or 2) is regarded as a necessary local condition for generation, it is likely in view of the symmetry condition (3.26), that dynamo action may be possible, i.e. the effect of generation is mirror symmetric about the equatorial plane. There is nothing profound in this statement as it is also implied by (3.25). However, the reflexional properties of A_{ij}, Γ_{ij} and $\lambda^{(1,2)}$ are of some interest.

4. $\alpha\omega$-Dynamos

In the previous section it was shown how the electric current $\boldsymbol{J}(\boldsymbol{x}, t)$ was likely to be able to sustain dynamo action. The effect of shearing the magnetic field lines was ignored. In a geomagnetic or stellar dynamo it is more likely that this effect is the principal mechanism for creating azimuthal magnetic field from the meridional magnetic field. It is manifest in the present model by supposing U and $\nabla(U/\rho)$ to be order 1 quantities. However, if both $\nabla(U/\rho)$ and Γ_{22} are order 1, the magnetic field will vary on the order 1 time scale. This paper is primarily concerned with variations on the (slow) diffusion time scale order R. Hence it is supposed that Γ_{22} is order R^{-1} (or possibly the time average (see § 5)) so that the ordering is not violated. Clearly the model considered in this section is the extreme opposite of that considered in § 3. It will become apparent that in general $\alpha\omega$-dynamos, varying on the order R time scale, are possible provided

$$\Gamma_{22}|\nabla(U/\rho)| = O(R^{-1}). \tag{4.1}$$

Equations governing the magnetic field are obtained on the assumption that

$$\Gamma_{22} = O(R^{-1}), \quad U = O(1), \quad \nabla(U/\rho) = O(1), \tag{4.2}$$

and that quantities vary on the order R time scale. It is apparent from the scaling that a dynamo model is likely with B order 1 and χ order R^{-1}. The notation is kept simple by setting

$$\left.\begin{aligned} \chi &= R^{-1}\chi^*, \\ \boldsymbol{b}_{\mathrm{M}} &= \nabla \times (\chi^* \boldsymbol{i}_\phi), \end{aligned}\right\} \tag{4.3}$$

and subsequently dropping the star. It follows that correct to lowest order (2.46), (2.47) become

$$R\frac{\partial\chi}{\partial t}+\frac{1}{\rho}\boldsymbol{u}_{\mathrm{M}}^{(+)}\cdot\nabla(\rho\chi) = R\Gamma_{22}B+\partial_i\left[\frac{1}{\rho}A_{ij}\,\partial_j(\rho\chi)\right]+\langle\boldsymbol{u}'\times\boldsymbol{b}'\rangle_\phi-[\mathscr{E}_1]_\phi, \tag{4.4a}$$

and
$$R\frac{\partial B}{\partial t}+\rho\nabla\cdot\left(\boldsymbol{u}_{\mathrm{M}}^{(-)}\frac{B}{\rho}\right) = \left(\nabla\frac{U}{\rho}\times\nabla\rho\chi\right)_\phi+\partial_i\left[\frac{1}{\rho}A_{ij}\,\partial_j(\rho B)\right]+O(R^{-1}), \tag{4.4b}$$

where $\boldsymbol{u}_{\mathrm{M}}^{(\pm)}$ are defined by (3.18). For future reference the vector

$$\Gamma_{\mathrm{M}i} = \Gamma_{2i} \quad (i = 1, 3) \tag{4.5}$$

is introduced, where $\boldsymbol{u}_{\mathrm{M}}^{(\pm)} = \boldsymbol{u}_{\mathrm{M}}\pm\boldsymbol{\Gamma}_{\mathrm{M}}\times\boldsymbol{i}_\phi$. Before (4.4) can be regarded as a deterministic pair of equations the averages $\langle\boldsymbol{u}'\times\boldsymbol{b}'\rangle_\phi$ and $[\mathscr{E}_1]_\phi$ must be obtained. An expression for $\boldsymbol{b}'$ is determined from (2.38) which to lowest order is

$$0 = \nabla\times\{\boldsymbol{u}'\times B\boldsymbol{i}_\phi+U\boldsymbol{i}_\phi\times\boldsymbol{b}'-\mathscr{E}'\}+O(R^{-1}), \tag{4.6}$$

where, from (2.39),
$$[\mathscr{E}']_i = \epsilon_{ip2}\alpha'_{kp}(\boldsymbol{X})\,\partial_k B-\mu'_{i2}(\boldsymbol{X})\,B+O(R^{-1}). \tag{4.7}$$

Evaluating the curls in (4.6), integrating with respect to ϕ and making use of the identity (2.33) leads to

$$\begin{aligned} b'_i = {}& \frac{B}{U}u'_i-\frac{1}{\rho U}\alpha'_{ik}(\boldsymbol{X})\,\partial_k(\rho B)+\delta_{i2}\frac{\rho}{U}\partial_j\left[\frac{1}{\rho}\hat{\alpha}'_{jk}(\boldsymbol{X})\,\partial_k(\rho B)\right]+\epsilon_{ijk}\frac{\rho}{U}\partial_j\,(\hat{\nu}'_k(\boldsymbol{X})\,B)+\frac{1}{U}\delta_{i3}\hat{\nu}'_2(\boldsymbol{X})\,B \\ & +\delta_{i2}\frac{\rho^2}{U}\left\{\hat{u}'_k\left[\frac{B}{U}\partial_k\left(\frac{U}{\rho}\right)-\partial_k\left(\frac{B}{\rho}\right)\right]-\left[\frac{1}{\rho U}\hat{\alpha}'_{jk}(\boldsymbol{X})\,\partial_j(\rho B)\right.\right. \\ & \left.\left.-\frac{\rho}{U}\epsilon_{kpq}\,\partial_p(\hat{\nu}'_q(\boldsymbol{X})\,B)-\frac{1}{U}\delta_{k3}\hat{\nu}'_2(\boldsymbol{X})\,B\right]\partial_k\left(\frac{U}{\rho}\right)\right\}+O(R^{-1}), \end{aligned} \tag{4.8}$$

where
$$\nu_i(\boldsymbol{X}) = \gamma_{2i}(\boldsymbol{X})-\tfrac{1}{2}\epsilon_{2ij}\,\partial_2[\alpha_{2j}(\boldsymbol{X})], \tag{4.9}$$

and the $\wedge$ operator is defined by

$$\partial\hat{\psi}'/\partial\phi = \psi' \quad\text{and}\quad \langle\hat{\psi}'\rangle = 0. \tag{4.10}$$

Evaluation of $\langle\boldsymbol{u}'\times\boldsymbol{b}'\rangle_\phi$ is now straightforward and gives

$$\langle\boldsymbol{u}'\times\boldsymbol{b}'\rangle_\phi = -\epsilon_{2ij}\frac{1}{U}\langle u'_i\{\alpha'_{jk}(\boldsymbol{X})\,(1/\rho)\,\partial_k(\rho B)-\epsilon_{jpq}\partial_p(\hat{\nu}'_q(\boldsymbol{X})\,\rho B)\}\rangle. \tag{4.11}$$

The reduction of $[\mathscr{E}_1]_\phi$ is more complicated. Using (2.39*b*) and (2.45), $[\mathscr{E}_1]_\phi$ is expressed in the form

$$[\mathscr{E}_1]_\phi = \epsilon_{2pq}\partial_k\langle\alpha'_{kp}(\boldsymbol{X})\,b'_q\rangle-\langle\nu'_j(\boldsymbol{X})\,b'_j\rangle. \tag{4.12}$$

The two terms on the right-hand side are evaluated separately and lead to

$$\begin{aligned} -\epsilon_{2pq}\partial_k\langle\alpha'_{kp}(\boldsymbol{X})\,b'_q\rangle = {}& -\epsilon_{2pq}\partial_k\left\{\langle\alpha'_{kp}(\boldsymbol{X})\,u'_q\rangle\frac{B}{U}\right\}+\partial_i\left\{\epsilon_{2pq}\partial_k\left[\frac{1}{\rho U}\langle\alpha'_{kp}(\boldsymbol{X})\,\alpha'_{iq}(\boldsymbol{X})\rangle\right]\rho B\right\} \\ & -\partial_k\left\langle\frac{1}{U}\epsilon_{2jm}\,\epsilon_{mpq}\,\alpha'_{kj}(\boldsymbol{X})\,\partial_p[\hat{\nu}'_q(\boldsymbol{X})\,\rho B]\right\rangle, \end{aligned} \tag{4.13}$$

and
$$\begin{aligned} \langle\nu'_j(\boldsymbol{X})\,b'_j\rangle = {}& \langle\nu'_j(\boldsymbol{X})\,u'_j\rangle\frac{B}{U}+\langle\nu'_2(\boldsymbol{X})\,\hat{u}'_k\rangle\frac{\rho^2}{U}\left[\frac{B}{U}\partial_k\left(\frac{U}{\rho}\right)-\partial_k\left(\frac{B}{\rho}\right)\right] \\ & +\epsilon_{2im}\,\epsilon_{mpq}\left\langle\nu'_p(\boldsymbol{X})\,\partial_q\left[\frac{1}{U}\hat{\alpha}'_{ik}(\boldsymbol{X})\,\partial_k(\rho B)\right]\right\rangle \\ & +\epsilon_{2jk}\left\{\frac{1}{\rho}\partial_j\left[\frac{\rho}{U}\langle\nu'_2(\boldsymbol{X})\,\hat{\nu}'_k(\boldsymbol{X})\rangle\rho B\right]+\frac{\rho}{U}\langle\nu'_2(\boldsymbol{X})\,\hat{\nu}'_k(\boldsymbol{X})\rangle\,\partial_j B\right. \\ & \left.-\frac{1}{\rho}\partial_j\left[\tfrac{1}{2}\rho\langle\hat{\nu}'_2(\boldsymbol{X})\,\hat{\nu}'_2(\boldsymbol{X})\rangle\,\partial_k\left(\frac{\rho}{U}\right)\rho B\right]-\tfrac{1}{2}\rho\langle\hat{\nu}'_2(\boldsymbol{X})\,\hat{\nu}'_2(\boldsymbol{X})\rangle\,\partial_k\left(\frac{\rho}{U}\right)\partial_j B\right\}. \end{aligned} \tag{4.14}$$

Collecting together the terms (4.10) to (4.14) gives an expression for the mean of the fluctuating terms in (4.4), namely

$$\langle \boldsymbol{u}' \times \boldsymbol{b}' \rangle_\phi - [\mathscr{E}_1]_\phi = QB - \tfrac{1}{2}\boldsymbol{P}\cdot\nabla B - \tfrac{1}{2}\nabla\cdot(\boldsymbol{P}B) + O(R^{-1}), \tag{4.15}$$

where
$$Q(\rho,z,t) = \epsilon_{2ij}\rho\,\partial_k\left[\frac{1}{\rho U}\langle u_i' \alpha_{jk}'(\boldsymbol{X})\rangle\right] + \frac{2}{U}\langle u_i' \nu_i'(\boldsymbol{X})\rangle - \frac{1}{U^2}\langle u_i'\partial_i[\rho U \hat{v}_2'(\boldsymbol{X})]\rangle, \tag{4.16a}$$

$$\boldsymbol{P}(\rho,z,t) = \boldsymbol{P}^{(1)} + \boldsymbol{P}^{(2)}, \tag{4.16b}$$

and
$$P_i^{(1)}(\rho,z,t) = -\epsilon_{2pq}\rho\,\partial_k[(1/\rho U)\{\langle \alpha_{pk}'(\boldsymbol{X})\,\alpha_{qi}'(\boldsymbol{X})\rangle + \delta_{pk}\delta_{qi}\rho^2\langle \hat{v}_2'(\boldsymbol{X})\,\hat{v}_2'(\boldsymbol{X})\rangle\}], \tag{4.16c}$$

$$P_i^{(2)}(\rho,z,t) = 2\epsilon_{mpq}\epsilon_{2jm}(\rho/U)\,\langle[\alpha_{ij}'(\boldsymbol{X}) + \rho\epsilon_{2ij}\,\hat{v}_2'(\boldsymbol{X})]\,\partial_p(\hat{v}_q'(\boldsymbol{X}))\rangle. \tag{4.16d}$$

The meridional components of $\boldsymbol{P}^{(1)}$ may also be expressed in the form

$$\boldsymbol{P}_{\mathrm{M}}^{(1)} = \rho^2\nabla\times([\psi/U]\,\boldsymbol{i}_\phi), \tag{4.17a}$$

where
$$\psi(\rho,z,t) = \tfrac{1}{2}\epsilon_{2pq}\epsilon_{2ij}\rho^{-2}\langle \alpha_{pi}'(\boldsymbol{X})\,\alpha_{qj}'(\boldsymbol{X})\rangle + \langle \hat{v}_2'(\boldsymbol{X})\,\hat{v}_2'(\boldsymbol{X})\rangle. \tag{4.17b}$$

Hence correct to lowest order (4.4) becomes

$$R\frac{\partial\chi}{\partial t} + \frac{1}{\rho}\boldsymbol{u}_{\mathrm{M}}^{(+)}\cdot\nabla(\rho\chi) = \Gamma B + [\boldsymbol{P}\times(\nabla\times B\boldsymbol{i}_\phi)]_\phi + \partial_i\left[\frac{1}{\rho}A_{ij}\partial_j(\rho\chi)\right] + O(R^{-1}), \tag{4.18a}$$

and
$$R\frac{\partial B}{\partial t} + \rho\nabla\cdot\left(\boldsymbol{u}_{\mathrm{M}}^{(-)}\frac{B}{\rho}\right)\left(\nabla\frac{U}{\rho}\times\nabla\rho\chi\right)_\phi + \partial_i\left[\frac{1}{\rho}A_{ij}\partial_j(\rho B)\right] + O(R^{-1}), \tag{4.18b}$$

where $\Gamma = R\Gamma_{22} + Q - \frac{1}{2}\rho^2\nabla\cdot\left(\frac{1}{\rho^2}\boldsymbol{P}^{(2)}\right)$. Hence (4.18) forms a closed pair of coupled equations characterized by the quantities $\boldsymbol{\Gamma}_{\mathrm{M}}$, $R\Gamma_{22}+Q$, $\boldsymbol{P}$ and A_{ij}.

The various terms in (4.18) are readily interpreted. Evidently (4.18*b*) is coupled to the meridional field only by the term $(\nabla U/\rho\times\nabla\rho\chi)_\phi$ so that shearing is the only mechanism which creates azimuthal magnetic field from the meridional magnetic field. The α-effect is different from that considered in § 3: the terms Γ_{11}, Γ_{13}, Γ_{33} are neglected and the coefficient of B in (4.18*a*) contains some new terms. The term $[\boldsymbol{P}\times(\nabla\times B\boldsymbol{i}_\phi)]_\phi$ in (4.18*a*) corresponds to the $\langle\boldsymbol{\omega}\rangle\times\langle\boldsymbol{j}\rangle$-effect discussed in § 1. An alternative interpretation can be given to the part corresponding to $\boldsymbol{P}^{(1)}$. Evidently it follows that

$$[\boldsymbol{P}^{(1)}\times(\nabla\times B\boldsymbol{i}_\phi)]_\phi = [\nabla(\rho\psi/U)\times\nabla(\rho B)]_\phi, \tag{4.19}$$

which is analogous to the shearing effect $(\nabla U/\rho\times\nabla\rho\chi)_\phi$. Thus this term may be regarded as a source of meridional magnetic field created by shearing the azimuthal magnetic field! Finally it will be shown in § 6 that, by a suitable choice of the transformation $\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x},t)$, $\boldsymbol{u}'$ may be made order R^{-1}. Hence without loss of generality it is supposed that

$$Q = O(R^{-1}). \tag{4.20}$$

and so is neglected subsequently.

As in § 3 a complete treatment of (4.18) is likely to be a formidable undertaking. Instead the approach used there is adopted again in order to help determine the significance of the various new terms in (4.18). It is supposed that

$$\boldsymbol{u}_{\mathrm{M}} = 0, \quad \rho\nabla U/\rho = \boldsymbol{\omega} \quad (= \text{constant}), \tag{4.21}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

and that $\boldsymbol{\Gamma}_M$, $R\Gamma_{22}$, $\boldsymbol{P}$ and A_{ij} are independent of position and time. In the planar geometry described by (3.4) solutions are sought proportional to $\exp(nt/R - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$ where $\boldsymbol{k}$ is given by (3.9). Making these substitutions in (4.18) leads to the pair of algebraic equations

$$\{n + k_i A_{ij} k_j + \mathrm{i}(\boldsymbol{\Gamma}_M \times \boldsymbol{k})_y\}\,\chi = \{R\Gamma_{22} + \mathrm{i}(\boldsymbol{P}\cdot\boldsymbol{k})\}\,B, \tag{4.22a}$$

$$\{n + k_i A_{ij} k_j - \mathrm{i}(\boldsymbol{\Gamma}_M \times \boldsymbol{k})_y\}\,B = -\mathrm{i}(\boldsymbol{\omega}\times\boldsymbol{k})_y\,\chi, \tag{4.22b}$$

and hence the dispersion relation

$$n = -k_i A_{ij} k_j \pm \{-(\boldsymbol{\omega}\times\boldsymbol{i}_y)\cdot\boldsymbol{k}(\boldsymbol{P}\cdot\boldsymbol{k}) - (\boldsymbol{\Gamma}_M\times\boldsymbol{k})^2 + \mathrm{i}R\Gamma_{22}(\boldsymbol{\omega}\times\boldsymbol{i}_y)\cdot\boldsymbol{k}\}^{\frac{1}{2}}, \tag{4.23}$$

is easily obtained. As in § 3, provided one of the square roots has a positive real part, dynamo action is possible for $\boldsymbol{k}$ sufficiently small.

Various cases may be considered. Case (*a*) $\boldsymbol{P} = \boldsymbol{\Gamma}_M = 0$; dynamo action is always possible. Case (*b*) $\boldsymbol{\Gamma}_M = R\Gamma_{22} = 0$; dynamo action is always possible provided $\boldsymbol{\omega}\times\boldsymbol{i}_y$ and $\boldsymbol{P}$ are not parallel. Case (*c*) $\boldsymbol{\omega}\boldsymbol{P} = R\Gamma_{22}\boldsymbol{\omega} = 0$; dynamo action is impossible. As in § 3 it appears that $\boldsymbol{\Gamma}_M$ may hinder dynamo action. Interpreting $-\boldsymbol{i}_\phi\times\boldsymbol{\Gamma}_M$ as a meridional velocity $\boldsymbol{u}_M^{(+)}$ advecting χ, it appears in (4.22*b*) as a meridional velocity $\boldsymbol{u}_M^{(-)}$ advecting B. Thus instead of just advecting the magnetic field, the meridional and azimuthal components of the magnetic field are advected in opposite directions which may be unfavourable for dynamo action. Case (*b*) suggests that the combination of the $\langle\boldsymbol{\omega}\rangle\times\langle\boldsymbol{j}\rangle$ and the ω-effect are capable of maintaining dynamos even in the absence of the α-effect. Indeed turbulent dynamo models of this type have been considered by Rädler (1968*a*, *b*, 1969*a*, *b*, 1970).

For the special case where the displacement vector is order $R^{-\frac{1}{2}}$:

$$R^{-\frac{1}{2}}\boldsymbol{\eta}(\boldsymbol{R},\boldsymbol{x},t) = \boldsymbol{X}(\boldsymbol{x},t) - \boldsymbol{x} = O(R^{-\frac{1}{2}}) \quad (\langle\boldsymbol{\eta}\rangle = O(R^{-\frac{1}{2}})), \tag{4.24}$$

it follows that

$$R\Gamma_{22} = O(1), \quad \boldsymbol{\Gamma}_M = O(R^{-1}), \quad \boldsymbol{P} = O(R^{-1}) \quad \text{and} \quad A_{ij} - \delta_{ij} = O(R^{-1}). \tag{4.25}$$

Thus (4.18) reduces to the equations obtained by Braginskiĭ (1964*a*): an exact correspondence is made in appendix D. The properties of the resulting equations have been examined extensively (see Braginskiĭ 1964*b*, *c*)). Braginskiĭ (1964*b*) made an analysis similar to that leading to the dispersion relation (4.23). However, the displacement vector $\boldsymbol{\eta}$ was given and the influence of boundaries considered. Thus a discrete spectrum was obtained and dynamo action is possible provided the generation coefficient is sufficiently large.

Care must be taken in distinguishing helicity, as measured by Γ_{22}, and knottedness of the magnetic field lines, as measured by

$$K = \int_{V_1} \boldsymbol{b}^*\cdot\boldsymbol{\chi}^*\,\mathrm{d}^3X,$$

where $\boldsymbol{b}^* = \nabla_X\times\boldsymbol{\chi}^*$ (see Moffatt 1969). For simplicity it is supposed that the fluid is surrounded by a perfect conductor and hence the normal component of the magnetic field vanishes on the boundary. For this case Woltjer (1958) has shown that K is conserved in a perfectly conducting fluid. However, in the presence of magnetic diffusion the modified result is

$$R\frac{\mathrm{d}K}{\mathrm{d}t} = -2\int \boldsymbol{b}^*\cdot\nabla_X\times\boldsymbol{b}^*\,\mathrm{d}^3X = -2\int\boldsymbol{b}\cdot\boldsymbol{\mathscr{E}}\,\mathrm{d}^3x. \tag{4.26}$$

If the magnetic field is alined to the loops $C(\boldsymbol{X})$ $(\boldsymbol{b} = B\boldsymbol{i}_\phi)$ the term on the right of (4.26) becomes

$$\int\boldsymbol{b}\cdot\boldsymbol{\mathscr{E}}\,\mathrm{d}^3x = 2\pi\int\Gamma_{22}B^2\rho\,\mathrm{d}\rho\,\mathrm{d}z. \tag{4.27}$$

In general (4.26) and (4.27) imply that K is non-zero contrary to the hypothesis that the magnetic field lines are not knotted, $K = 0$. When $\boldsymbol{b} = B\boldsymbol{i}_\phi + O(R^{-1})$, as assumed throughout this section, the knottedness K is order R^{-1} and this is consistent with (4.26) provided Γ_{22} is order R^{-1} also. Since non-zero Γ_{22} is an essential ingredient of the Braginskiĭ dynamo it must be emphasized that the topology of the loops $C(\boldsymbol{X})$ does not imply that $\Gamma_{22} = 0$. Indeed an explicit example of a flow with non-zero Γ_{22} is provided by Braginskiĭ (1964*c*). It is interesting to compare the above remarks with the small magnetic Reynolds number dynamo considered by Moffatt (1970*a*) for which knottedness is essential for a non-zero α-effect in pseudo-isotropic turbulence.

5. The hydromagnetic dynamo

In the previous sections the kinematic dynamo theory has been developed on the assumption that there is only one length scale and one time scale (the free decay time). In the context of the Earth's dynamo Braginskii (1967) has emphasized the possible importance of MAC-waves, so called since a significant role is played by the magnetic, buoyancy (Archimedean), and Coriolis forces. These waves fluctuate on a time scale which is short compared to the free decay time. Some of the difficulties that arise in the dynamo theory involving such waves can only be appreciated by studying the magnetic induction equation in conjunction with the equation of motion. Consequently the hydromagnetic dynamo problem will now be considered.

In dimensionless form the equation of motion is

$$R_0\{\partial \boldsymbol{u}^*/\partial t + (\nabla_{\boldsymbol{X}} \times \boldsymbol{u}^*) \times \boldsymbol{u}^*\} + \boldsymbol{i}_z \times \boldsymbol{u}^* = -\nabla_{\boldsymbol{X}} p^* + \boldsymbol{j}^* \times \boldsymbol{b}^* + \boldsymbol{F}^*, \tag{5.1}$$

(see Tough & Roberts 1968) where $R_0 = \nu^*/2\Omega L$ is the Rossby number, $\Omega \boldsymbol{i}_z$ is the rotation vector, and $\boldsymbol{F}^*$ is the body force. The scaling of the dimensionless magnetic field and body force in (5.1) has been chosen so that the Lorenz and body forces are comparable with the Coriolis force. Viscous forces are likely to be negligible except in boundary layers and so have been omitted from the present considerations. Braginskiĭ (1967) supposed that a significant contribution to $\boldsymbol{F}^*$ is provided by density variations (within the framework of the Boussinesq approximation). This part of the force $\boldsymbol{F}^*_A$ (say) is given by

$$\boldsymbol{F}^*_A = \theta^* \boldsymbol{g}, \tag{5.2}$$

where $\boldsymbol{g}$ is the acceleration due to gravity and θ^* is proportional to the excess density. If it is supposed that the variation in density is caused by the temperature then, in the absence of heat sources, θ^* satisfies the heat conduction equation

$$\frac{\partial \theta^*}{\partial t} + \boldsymbol{u}^* \cdot \nabla_{\boldsymbol{X}} \theta^* = \left(\frac{k\mu\sigma}{R}\right) \nabla^2_{\boldsymbol{X}} \theta^*, \tag{5.3}$$

where k is the thermal diffusivity. After making the transformation (1.20), equations (5.1) to (5.3) become

$$R_0\left\{\frac{\partial \boldsymbol{G}}{\partial t} + (\nabla \times \boldsymbol{G}) \times \boldsymbol{u}\right\} + \boldsymbol{C} = -\nabla P + (\nabla \times \boldsymbol{H}) \times \boldsymbol{b} + \boldsymbol{F}, \tag{5.4†}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta = \left(\frac{k\mu\sigma}{R}\right) \frac{1}{\rho} \partial_i[\rho \alpha_{ij}(\boldsymbol{X})\, \partial_j \theta], \tag{5.5}$$

† Equation (5.4) has some similarity with the material form of the equation of motion in a Lagrangian framework (see, for example, Serrin 1959, eqn (6.10) and §29). The ith component of (5.4) is obtained after contraction of the jth component of (5.1) with $a_{ji}(\boldsymbol{x})$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where
$$\left.\begin{aligned} G_i(\boldsymbol{x},t) &= a_{mi}(\boldsymbol{X})\,u^*_m(\boldsymbol{X},t),\\ P(\boldsymbol{x},t) &= P^*(\boldsymbol{X},t)-R_0(\partial\boldsymbol{X}/\partial t)\cdot\boldsymbol{u}^*(\boldsymbol{X},t),\\ C_i(\boldsymbol{x},t) &= \epsilon_{m3k}\,a_{mi}(\boldsymbol{X})\,u^*_k(\boldsymbol{X},t),\\ H_i(\boldsymbol{x},t) &= a_{mi}(\boldsymbol{X})\,b^*_m(\boldsymbol{X},t),\\ F_i(\boldsymbol{x},t) &= a_{mi}(\boldsymbol{X})\,F^*_m(\boldsymbol{X},t),\\ \theta(\boldsymbol{x},t) &= \theta^*(\boldsymbol{X},t),\end{aligned}\right\} \tag{5.6}$$
and
$$F_{Ai}(\boldsymbol{x},t) = \theta(\boldsymbol{x},t)\,a_{mi}(\boldsymbol{X})\,g_m(\boldsymbol{X}). \tag{5.7}$$

Since there is no reason to suppose that $\nabla U/\rho$ is small, the scaling adopted for the magnetic field is the same as in the previous section (see (4.3)). Thus neglecting the inertia terms ($R_0 = 0$), and supposing the displacement $\boldsymbol{X}-\boldsymbol{x}$ is order $R^{-\frac{1}{2}}$ it is clear that the ϕ-average of (5.4) gives

$$\frac{\partial}{\partial z}(U-B^2/\rho) = -(\nabla\times\langle\boldsymbol{F}\rangle)_\phi + O(R^{-\frac{1}{2}}), \tag{5.8a}$$

$$R\frac{\partial}{\partial t}\left\{\frac{1}{2}\left\langle\boldsymbol{X}\times\frac{1}{\rho}\frac{\partial\boldsymbol{X}}{\partial\phi}\right\rangle_z\right\}+u_\rho = \frac{1}{\rho}(\boldsymbol{b}_{\mathrm{M}}\cdot\nabla)\,\rho B + R\langle F_\phi\rangle + O(R^{-\frac{1}{2}}), \tag{5.8b}$$

where
$$\langle F_\phi\rangle = \frac{1}{2\pi\rho}\oint_{C(X)}\boldsymbol{F}^*\cdot\mathrm{d}\boldsymbol{X}. \tag{5.9}$$

Indeed the error terms are order R^{-1}, if $\langle\boldsymbol{X}-\boldsymbol{x}\rangle$ is order R^{-1}. The equations (5.8) are in agreement with those determined by Tough & Roberts (1968) where $\boldsymbol{u}_{\mathrm{M}}$ and $\boldsymbol{b}_{\mathrm{M}}$ are replaced by effective variables. The representation for $\langle F_\phi\rangle$ is new. Its striking simplicity (cf. Tough & Roberts 1968, eqn (46) and Soward 1971*a*, eqn (21)) is comparable to the expression (2.52) for Γ_{22}. Now the ϕ-component of the Coriolis force can be represented in the alternative form

$$R\frac{\partial}{\partial t}\left\{\frac{1}{2}\left\langle\boldsymbol{X}\times\frac{1}{\rho}\frac{\partial\boldsymbol{X}}{\partial\phi}\right\rangle_z\right\} = R^{\frac{1}{2}}\frac{\partial}{\partial t}\langle\eta_\rho\rangle+\frac{\partial}{\partial t}\left\{\frac{1}{2}\left\langle\boldsymbol{\eta}\times\frac{1}{\rho}\frac{\partial\boldsymbol{\eta}}{\partial\phi}\right\rangle_z\right\}, \tag{5.10}$$

where $R^{-\frac{1}{2}}\boldsymbol{\eta} = \boldsymbol{X}-\boldsymbol{x}$. Thus provided $R\langle F_\phi\rangle$ is order 1, $\partial\langle\eta_\rho\rangle/\partial t$ is necessarily order $R^{-\frac{1}{2}}$. Moreover, assuming $k\mu\sigma$ is order 1 it follows that θ is predominantly axisymmetric, we have

$$\theta(\boldsymbol{x},t) = \theta_0(\rho,z,t)+O(R^{-1}) \tag{5.11}$$

(provided that θ_0 varies on the slow time scale) or that $\theta^*(\boldsymbol{X},t)$ is approximately constant on the loops $C(\boldsymbol{X})$. Since the gravitational force is conservative ($\nabla_X\times\boldsymbol{g} = 0$) it follows from (5.9) that $\langle F_{A\phi}\rangle$ is order R^{-1}. This result is compatible with the scaling that has been adopted for the body forces in (5.8).

A perturbation procedure is now outlined for considering MAC-waves. As before the inertia terms are neglected ($R_0 = 0$). This approximation is equivalent to the neglect of the inertial waves. These waves have a short time scale order R_0 and are unlikely to be relevant in the present dynamo model. However, they may provide a mechanism for dynamo maintenance in the multiple length scale models (see Moffatt 1970*b*). It is supposed that the body force is due only to buoyancy ($\boldsymbol{F}^* = \boldsymbol{F}^*_A$) and that in the lowest order approximations the flow and magnetic field are axisymmetric and azimuthal ($\boldsymbol{u} = U\boldsymbol{i}_\phi$, $\boldsymbol{b} = B\boldsymbol{i}_\phi$). Thus the curl of (5.4) gives

$$\left(\frac{\partial}{\partial t}+\frac{U}{\rho}\frac{\partial}{\partial\phi}\right)[T_{3i}(\boldsymbol{X})]-\delta_{i2}T_{3j}(\boldsymbol{X})\,\rho\,\partial_j\left(\frac{U}{\rho}\right) = \frac{B}{\rho}\frac{\partial}{\partial\phi}[(\nabla\times\boldsymbol{H})_i]-\delta_{i2}\rho(\nabla\times\boldsymbol{H})\cdot\nabla\frac{B}{\rho}+(\nabla\times\boldsymbol{F}_A)_i, \tag{5.12}$$

Setting $R^{-\frac{1}{2}}\boldsymbol{\eta} = \boldsymbol{X}(\boldsymbol{x}, t) - \boldsymbol{x}$ it follows that

$$T_{3i}(\boldsymbol{X}) = [\boldsymbol{i}_z - R^{-\frac{1}{2}}\,\partial\boldsymbol{\eta}/\partial z + O(R^{-1})]_i, \tag{5.13a}$$

$$\boldsymbol{H}(\boldsymbol{x}, t) = B[\boldsymbol{i}_\phi + R^{-\frac{1}{2}}\,(1/\rho)\,\{\partial\boldsymbol{\eta}/\partial\phi + \boldsymbol{i}_z\times\boldsymbol{\eta} + \nabla(\rho\eta_\phi)\} + O(R^{-1})], \tag{5.13b}$$

$$\boldsymbol{F}_A(\boldsymbol{x}, t) = \theta\boldsymbol{g} + R^{-\frac{1}{2}}\{\nabla(\theta\boldsymbol{\eta}\cdot\boldsymbol{g}) - \boldsymbol{g}(\boldsymbol{\eta}\cdot\nabla\theta) - \boldsymbol{\eta}\times(\nabla\times\theta\boldsymbol{g})\} + O(R^{-1}), \tag{5.13c}$$

and

$$\nabla\cdot\boldsymbol{\eta} = O(R^{-\frac{1}{2}}). \tag{5.14}$$

The expression for $\boldsymbol{F}_A$ is readily derived from the identity

$$F_{Ai} = \theta[g_i + R^{-\frac{1}{2}}\{a_{im}(\boldsymbol{g})\,\eta_m + g_m a_{mi}(\boldsymbol{\eta})\} + O(R^{-1})].$$

The first approximation of (5.12) just gives (5.8 *a*). The second approximation gives the linearized wave equation for the displacement vector $\boldsymbol{\eta}$ which must be solved subject to the condition that the normal component of $\boldsymbol{\eta}$ vanishes on the boundary. The order $R^{-\frac{1}{2}}$ term involving $(\nabla\times\theta\boldsymbol{g})$ in (5.13 *c*) may be expressed in terms of U and B by using the first approximation of (5.12). Thus it can be shown that the wave equation given here corresponds to Braginskiĭ (1967) equation (2.14).

Solutions of (5.12) may be sought in the form

$$\boldsymbol{\eta} = \sum_{\omega, m} \boldsymbol{\eta}_{\omega, m}(\rho, z)\,\mathrm{e}^{\mathrm{i}(\omega t - m\phi)} + \sum_{\omega, m} \boldsymbol{\eta}^*_{\omega, m}(\rho, z)\,\mathrm{e}^{-\mathrm{i}(\omega t - m\phi)}, \tag{5.15}$$

where here the star denotes the complex conjugate and m is an integer. For clarity the dependence of $\boldsymbol{\eta}$ on the slow time scale has been suppressed in (5.15). Evidently solutions may exist with ω complex corresponding to an exponential growth of $\boldsymbol{\eta}$. However, it will be assumed that conditions are such that the waves are stable (ω real). It is further assumed that given ω and m that $\boldsymbol{\eta}_{\omega, m}$ is defined uniquely except for a constant of proportionality.

The question may be posed: will waves of the type (5.15) help to sustain a dynamo? Evidently the analysis of the previous section must be extended to cope with the fast (order 1) time scale on which these waves fluctuate. Since the temporal development of B and χ on the slow (order R) time scale is the principal interest, a time average of (4.4) is taken. The time average of a quantity f is denoted by $\langle f\rangle^t$. The precise definition of the averaging operation need not be specified. Though of course $\langle f\rangle^t$ must vary on the slow time scale. Finally for the present analysis it is unnecessary to impose the restriction $\langle\boldsymbol{\eta}\rangle = O(R^{-1})$ on the displacement vector. Instead it is required that

$$\langle\langle\boldsymbol{\eta}\rangle\rangle^t = O(R^{-1}). \tag{5.16}$$

Hence axisymmetric oscillations are not excluded from the analysis. Since $R\Gamma_{22}$ is order 1, it follows immediately from (4.4) that $\chi - \langle\chi\rangle^t$ and $B - \langle B\rangle^t$ are order R^{-1}. Hence the time average of (4.4) gives

$$R\frac{\partial}{\partial t}\langle\chi\rangle^t + \frac{1}{\rho}\langle\boldsymbol{u}_{\mathrm{M}}\rangle^t\cdot\nabla(\rho\langle\chi\rangle^t) = R\langle\Gamma_{22}\rangle^t\langle B\rangle^t + \left(\nabla^2 - \frac{1}{\rho^2}\right)\langle\chi\rangle^t + O(R^{-1}), \tag{5.17a}$$

$$R\frac{\partial}{\partial t}\langle B\rangle^t + \rho\langle u_{\mathrm{M}}\rangle^t\cdot\nabla\left(\frac{\langle B\rangle^t}{\rho}\right) = \left\{\nabla\left(\frac{\langle U\rangle^t}{\rho}\right)\times\nabla(\rho\langle\chi\rangle^t)\right\}_\phi + \left(\nabla^2 - \frac{1}{\rho^2}\right)\langle B\rangle^t + O(R^{-1}). \tag{5.17b}$$

The generation term may now be determined from (5.15) and (C 11) where the double average of $\mathrm{e}^{\mathrm{i}(\omega t - m\phi)}$ is defined by

$$\langle\langle\mathrm{e}^{\mathrm{i}(\omega t - m\phi)}\rangle\rangle^t = \begin{cases} 1 & \text{if} \quad \omega = 0,\ m = 0, \\ 0 & \text{otherwise.}\end{cases} \tag{5.18}$$

However, it may be shown from (5.13) and (5.14) that $\boldsymbol{\eta}_{\omega,m}(\rho, z)$ can be expressed in the form

$$\boldsymbol{\eta}_{\omega,m}(\rho, z) = \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\alpha_{\omega,m}}\{\eta'_{\omega,m:\rho}, -\mathrm{i}\eta'_{\omega,m:\phi}, \eta'_{\omega,m:z}\}, \tag{5.19}$$

where the primed quantities are real functions of ρ and z. Hence (5.15) may be expressed in the form

$$\boldsymbol{\eta} = \sum_{\omega,m}\{\eta'_{\omega,m:\rho}\cos(\omega t - m\phi - \alpha_{\omega,m}), \eta'_{\omega,m:\phi}\sin(\omega t - m\phi - \alpha_{\omega,m}), \eta'_{\omega,m:z}\cos(\omega t - m\phi - \alpha_{\omega,m})\} \tag{5.20}$$

It follows immediately that

$$R\langle\Gamma_{22}\rangle^t = 0. \tag{5.21}$$

This difficulty was appreciated by Braginskiĭ (1964*b*) in his original paper concerning dynamo action on the fast time scale. It is likely that solutions of (5.12) exist such that $R\langle\Gamma_{22}\rangle^t$ is non-zero, but these solutions will not satisfy the boundary conditions. The situation is similar to that considered by Moffatt (1970*b*) for random inertial waves. Moffatt notes that non-zero helicity requires a net energy flux (positive or negative) in the direction of the rotation vector. Clearly, for the problem considered here, there can be no net energy flux as there are no sources or sinks for the energy. Boundary-layer dissipation may contribute to a net energy flux but this will clearly only give rise to a small $R\langle\Gamma_{22}\rangle^t$. Alternatively, the displacement vector may be larger than order $R^{-\frac{1}{2}}$. If the displacement vector is order $R^{-\frac{1}{4}}$, (5.17) is still valid with the error terms now order $R^{-\frac{1}{2}}$. However, (5.12) must be considered up to the third approximation. It is likely that the distortion of the waves by nonlinear interaction is sufficient to provide order 1 generation, i.e. $R\langle\Gamma_{22}\rangle^t = O(1)$.

The equations (5.17) are evidently more general in their applicability than those proposed by Braginskiĭ (1964*b*). First, Braginskiĭ's analysis requires that $\boldsymbol{\eta}$ is represented by the sum of waves and oscillations. Though this decomposition is clearly correct if a well-ordered wave motion is considered, the present analysis is not restricted by such considerations. The difference is due to the fact that Braginskiĭ was obliged to solve the induction equation to obtain the fluctuating magnetic field before the generation term appeared from the analysis. Secondly, the equations (5.17) are valid for any small displacement and are not restricted by the requirement that the displacement is order $R^{-\frac{1}{2}}$. However, if the displacement is order $\epsilon(1 \gg \epsilon \gg R^{-\frac{1}{2}})$ then the error terms in (5.17) are order ϵ^2 and not order R^{-1}.

6. The coordinate transformation

The coordinate transformation $\boldsymbol{X}(\boldsymbol{x}, t)$ is not uniquely defined for two reasons. First, given a set of loops $C(\boldsymbol{X})$ any new transformation

$$\boldsymbol{X}^{(N)} = \boldsymbol{X}(\boldsymbol{X}_0(\boldsymbol{x}), t), \tag{6.1}$$

where $\boldsymbol{X}_0(\boldsymbol{x})$ is a coordinate transformation mapping the circles ρ = const., z = const. into the circles ρ_0 = const., z_0 = const. (of course it must also satisfy the requirements of the transformation $\boldsymbol{X}(\boldsymbol{x}, t)$), describes the same set of loops $C(\boldsymbol{X})$. Secondly, owing to the restriction (ii) in § 2 (*b*) the position of the loops in space is only uniquely defined to lowest order. Specifically, given a transformation $\boldsymbol{X}(\boldsymbol{x}, t)$ then constraint (ii) is still satisfied by the new transformation

$$\boldsymbol{X}^{(N)} = \boldsymbol{X}(\boldsymbol{x}, t) + R^{-1}\boldsymbol{\eta}^*(\boldsymbol{X}, t), \tag{6.2}$$

since the small change only effects the order R^{-1} part of the related velocity.

For models such as those proposed by Braginskiĭ (1964*a*, *b*) where the deviation of the loops $C(\boldsymbol{X})$ from circles is small, the first difficulty is not serious. Evidently it is natural to set

$$\boldsymbol{X}(\boldsymbol{x},t) = \boldsymbol{x}+\epsilon\boldsymbol{\eta}(\boldsymbol{x},t) \quad (\epsilon \ll 1). \tag{6.3}$$

The explicit choice of $\boldsymbol{\eta}$ is resolved by considerations similar to those leading to a choice of $\boldsymbol{\eta}^*$ in (6.2). For a finite amplitude displacement the best choice of the transformation is not clear. It would seem natural to identify a loop $C(\boldsymbol{X})$ with a circle which is located at ρ = const., z = const. where (ρ, z) is the mean position (in some sense) of the meridional components of $\boldsymbol{X}$. Consider the mapping

$$(\rho, z) \to (\langle T_{l1}(\boldsymbol{X})\, X_l\rangle, \langle T_{l3}(\boldsymbol{X})\, X_l\rangle). \tag{6.4}$$

Since (2.15) gives

$$(1/\rho)\,\partial_i\{\rho\langle T_{li}(\boldsymbol{X})\, X_l\rangle\} = 3, \tag{6.5}$$

the mapping preserves volume. Moreover, for the trivial case where the transformation $\boldsymbol{X}(\boldsymbol{x},t)$ maps circles into circles, the choice $\boldsymbol{X} = \boldsymbol{x}$ maps (ρ, z) into itself by (6.4). These two considerations suggest that a suitable averaging criterion would be to require that the mapping (6.4) is the identity transformation. The author has been unable to determine whether this criterion can be satisfied in general. Clearly the transformation does not have to satisfy such a criterion: the actual choice is directed by expediency.

In order to determine the effect of small variations in the shape of the loops $C(\boldsymbol{X})$ the related velocity $\boldsymbol{u}^{(N)}(\boldsymbol{x},t)$ is determined for the new transformation (6.2) in terms of the original related velocity $u(\boldsymbol{x},t)$. From (1.34) the velocity at $\boldsymbol{X}^{(N)}$ is given by

$$u_i^*(\boldsymbol{X}^{(N)},t) = \left(\frac{\partial \boldsymbol{X}^{(N)}}{\partial t}\right)_i + a_{im}(\boldsymbol{X}^{(N)})\, u_m^{(N)}(\boldsymbol{x},t). \tag{6.6}$$

Hence by expanding in a Taylor series at $\boldsymbol{X}^{(N)}$ the velocity at $\boldsymbol{X}$ is

$$\boldsymbol{u}^*(X,t) = \boldsymbol{u}^*(\boldsymbol{X}^{(N)},t) - R^{-1}\boldsymbol{\eta}^*\cdot\nabla_{X^{(N)}}\boldsymbol{u}^*(\boldsymbol{X}^{(N)},t) + O(R^{-2}). \tag{6.7}$$

Now since the difference $\boldsymbol{u}^{(N)}(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t)$ is order R^{-1} it follows from (1.34) and (6.7) that

$$Ra_{im}(\boldsymbol{X})\,(u_m^{(N)} - u_m) + [(\partial\boldsymbol{\eta}^*/\partial t)_i + a_{im}(\boldsymbol{\eta}^*)\, u_m] - [\boldsymbol{\eta}^*\cdot\nabla_X\boldsymbol{u}^*(X,t)]_i = O(R^{-1}). \tag{6.8}$$

Contracting with $T_{ij}(\boldsymbol{X})$ and making use of the basic identities in § 2 (*a*) gives

$$\boldsymbol{u}^{(N)}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) - R^{-1}\{\partial\boldsymbol{\eta}/\partial t - \nabla\times(\boldsymbol{u}\times\boldsymbol{\eta})\} + O(R^{-2}), \tag{6.9}$$

where

$$\eta_i^*(\boldsymbol{X},t) = a_{im}(\boldsymbol{X})\,\eta_m(\boldsymbol{x},t), \tag{6.10}$$

and

$$\nabla\cdot\boldsymbol{\eta} = O(R^{-1}). \tag{6.11}$$

The divergence of $\boldsymbol{\eta}$ vanishes to lowest order as a result of the condition $J(\boldsymbol{X}^{(N)}) = 1$. Since $\boldsymbol{u}(\boldsymbol{x},t)$ is predominantly azimuthal it is more convenient to express (6.9) in the form

$$u_i^{(N)} = U\delta_{i2} + R^{-1}\left[u_{Mi} + u_i' - \left\{\frac{\partial\eta_i}{\partial t} + \frac{U}{\rho}\frac{\partial\eta_i}{\partial\phi} - \delta_{i2}\rho(\boldsymbol{\eta}\cdot\nabla)\frac{U}{\rho}\right\}\right] + O(R^{-2}). \tag{6.12}$$

For the slow time scale it is clear that, if $\boldsymbol{\eta}$ is defined by

$$\eta_i = (\rho/U)\,\hat{u}_i' - \delta_{i2}\,\hat{u}_k'\rho\partial_k(\rho/U) + O(R^{-1}), \tag{6.13}$$

the related velocity $\boldsymbol{u}^{(N)}$ is

$$\boldsymbol{u}^{(N)}(\boldsymbol{x},t) = U(\rho,z,t)\,\boldsymbol{i}_\phi + R^{-1}\boldsymbol{u}_{\mathrm{M}}(\rho,z,t) + O(R^{-2}). \tag{6.14}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Consequently, corresponding to the new transformation the fluctuating related velocity appears only at order R^{-2}.

Examination of the mean equations (3.3) and (4.18) shows that they are insensitive to any order R^{-1} displacement of the loops $C(X)$. This result is trivial except for the coefficient $R\Gamma_{22}+Q$ appearing in (4.18*a*). The invariance is established (on the slow time scale) by showing that

$$R\Gamma_{22}^{(N)} = R\Gamma_{22}+Q+O(R^{-1}), \tag{6.15}$$

where $\Gamma_{22}^{(N)}$ is evaluated for the transformation $X^{(N)}(x,t)$ defined by (6.2) and (6.13). Noting the identities

$$I_1 = -\left\langle \frac{1}{\rho}\frac{\partial X}{\partial \phi}\cdot\nabla_X\times\left\{\frac{1}{\rho}\frac{\partial \eta^*}{\partial \phi}-\eta^*\cdot\nabla_X\left(\frac{1}{\rho}\frac{\partial X}{\partial \phi}\right)\right\}\right\rangle,$$
$$= -\langle \epsilon_{2ij}\,\partial_k[\alpha'_{ik}(X)\,\partial_2\eta_j]\rangle+\langle \nu'_j(X)\,\partial_2\eta_j\rangle+\left\langle\frac{1}{\rho}\nu'_2(X)\,\eta_1\right\rangle, \tag{6.16a}$$

$$I_2 = -\left\langle \left\{\frac{1}{\rho}\frac{\partial \eta^*}{\partial \phi}-\eta^*\cdot\nabla_X\left(\frac{1}{\rho}\frac{\partial X}{\partial \phi}\right)\right\}\cdot\nabla_X\times\left(\frac{1}{\rho}\frac{\partial X}{\partial \phi}\right)\right\rangle,$$
$$= \left\langle \epsilon_{2ij}\frac{1}{\rho}\alpha'_{i1}(X)\,\partial_2\eta_j\right\rangle+\langle \nu'_j(X)\,\partial_2\eta_j\rangle+\left\langle\frac{1}{\rho}\nu'_2(X)\,\eta_1\right\rangle, \tag{6.16b}$$

$$I_3 = -\left\langle \eta^*\cdot\nabla_X\left\{\frac{1}{\rho}\frac{\partial X}{\partial \phi}\cdot\nabla_X\times\left(\frac{1}{\rho}\frac{\partial X}{\partial \phi}\right)\right\}\right\rangle,$$
$$= \langle\{\partial_j\,\nu'_2(X)\}\,\eta_j\rangle, \tag{6.16c}$$

obtained with the help of (2.31*b*), it follows that

$$R\Gamma_{22}^{(N)} = -R\oint\frac{1}{\rho}\frac{\partial}{\partial\phi}(X+R^{-1}\eta^*)\cdot\left\{\nabla_{X+R^{-1}\eta^*}\times\frac{1}{\rho}\frac{\partial}{\partial\phi}(X+R^{-1}\eta^*)\right\}\mathrm{d}\phi,$$
$$= R\Gamma_{22}+(I_1+I_2+I_3)+O(R^{-1}). \tag{6.17}$$

Moreoever, by using (6.13) it can be shown that

$$Q = I_1+I_2+I_3, \tag{6.18}$$

and hence (6.15) is established. In view of these remarks the scalar $R\Gamma_{22}+Q$ should be regarded as a single quantity: the independent values of $R\Gamma_{22}$ and Q are unimportant.

One final consideration should be given to the choice of the transformation $X(x,t)$, namely the boundary conditions. The earlier analysis suggests that boundary conditions may be applied without much difficulty if $X(x,t) = x$ on and near the boundary. If the mean equations (3.3) and (4.17) are valid up to the boundary, as is the case if the azimuthal velocity remains large, then the boundary conditions become $B = 0$ and χ is continuous where in the exterior region $\chi(\rho,z,t)\,\boldsymbol{i}_\phi$ is the magnetic vector potential. On the other hand, only the normal component of the fluctuating magnetic field is continuous: the tangential component adjusts in a boundary layer of thickness order R^{-1} (cf. Braginskiĭ 1964*a*, § 4).

7. Discussion

The principal achievement of this paper has been the determination of equations (3.3) and (4.18). It is unfortunate that evaluation of the coefficients, which characterize the equations, is so difficult. However, their form and the elementary arguments of §§ 3 and 4 do suggest that

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

the equations govern dynamo models. It seems likely that the analysis may be relevant to the Earth's dynamo; especially the treatment of $\alpha\omega$-dynamos in § 4. In any case the analysis gives insight into possible dynamo mechanisms that may be important for large magnetic Reynolds number flows. The necessity of the product $\Gamma_{22}|\nabla U/\rho|$ being order R^{-1} emphasizes the importance of considering both the equation of motion and the magnetic induction equation simultaneously. Evidently if the criterion is violated the magnetic field may grow very rapidly. However, the increased magnetic field strength inhibits the fluid motion and consequently the magnitude of the generation mechanism is decreased.

Finally though Braginskiĭ's (1964*a*, *b*) dynamo equations have been obtained here as a special case, the importance of the new derivation and the more extensive applicability should not be overlooked. Indeed this last consideration provided the original motivation for the analysis.

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Appendix A

The identity (2.33), namely

$$(1/\rho)\,\partial_j\{\rho\alpha_{ij}(\boldsymbol{X})\} = \epsilon_{ipq}\mu_{pq}(\boldsymbol{X}), \tag{A 1}$$

is established and the expression for $\mathscr{E}$ (2.29), namely

$$\mathscr{E}_i(\boldsymbol{x},t) = \epsilon_{ipq}\alpha_{kp}(\boldsymbol{X})\,\partial_k b_q - \mu_{ij}(\boldsymbol{X})\,b_j, \tag{A 2}$$

is derived from the definition (2.28).

Contracting $\mu_{pq}(\boldsymbol{X})$ (defined by (2.30*b*)) with ϵ_{ipq} gives

$$\epsilon_{ipq}\mu_{pq}(\boldsymbol{X}) = T_{pj}(\boldsymbol{X})\,\partial_j[T_{pi}(\boldsymbol{X})] - T_{pi}(\boldsymbol{X})\,\partial_j[T_{pj}(\boldsymbol{X})] \\ -(1/\rho)\,\{2\epsilon_{3pq}T_{p2}(\boldsymbol{X})\,T_{qi}(\boldsymbol{X}) + T_{pi}(\boldsymbol{X})\,T_{p1}(\boldsymbol{X})\}, \tag{A 3}$$

while differentiation of $\alpha_{ij}(\boldsymbol{X})$ (defined by (2.30*a*)) gives

$$(1/\rho)\,\partial_j\{\rho\alpha_{ij}(\boldsymbol{X})\} = T_{pj}(\boldsymbol{X})\,\partial_j[T_{pi}(\boldsymbol{X})] + T_{pi}(\boldsymbol{X})\,\partial_j[T_{pj}(\boldsymbol{X})] + (1/\rho)\,T_{pi}(\boldsymbol{X})\,T_{p1}(\boldsymbol{X}). \tag{A 4}$$

Now, by using (2.14*c*), (A 1) follows immediately.

With the help of (2.6) and (1.18) $\mathscr{E}$ (defined by (2.28)) is given in the form

$$\mathscr{E}_i(\boldsymbol{x},t) = \epsilon_{jpq}a_{ji}(\boldsymbol{X})\,T_{pl}(\boldsymbol{X})\,\{\partial_l[a_{qm}(\boldsymbol{X})\,b_m] + \delta_{l2}\epsilon_{3nq}(1/\rho)\,a_{nm}(\boldsymbol{X})\,b_m\}. \tag{A 5}$$

The differentiation of the first term is carried out and use of (2.10*b*) and (2.14*a*) leads to

$$\mathscr{E}_i(\boldsymbol{x},t) = \epsilon_{inm}\alpha_{nl}(\boldsymbol{X})\,\partial_l b_m + \epsilon_{jpq}a_{ji}(\boldsymbol{X})\,T_{pl}(\boldsymbol{X})\,\{\partial_m[a_{ql}(\boldsymbol{X})] - \epsilon_{3lm}(1/\rho)\,a_{q2}(\boldsymbol{X}) \\ +(1/\rho)\,\epsilon_{3nq}\delta_{m2}a_{nl}(\boldsymbol{X})\}\,b_m. \tag{A 6}$$

Since by (1.32) $T_{pl}(\boldsymbol{X})\,\partial_m[a_{ql}(\boldsymbol{X})] = -a_{ql}(\boldsymbol{X})\,\partial_m[T_{pl}(\boldsymbol{X})]$, it is now a straightforward matter to verify (A 2).

Appendix B

The identities (2.41), (2.42), namely

$$-[\mathscr{E}_0]_\phi = \partial_i[(1/\rho)\,A_{ij}\,\partial_j(\rho\chi)] + \Gamma_{2j}B_j, \tag{B 1}$$

$$-[\nabla\times\mathscr{E}_0]_\phi = \partial_i[(1/\rho)\,A_{ij}\,\partial_j(\rho B)] + \epsilon_{2ij}\,\partial_i(\Gamma_{jk}B_k) \tag{B 2}$$

are established, where Γ_{ij} (defined by (2.44) and (2.45)) is the symmetric matrix

$$\Gamma_{ij}(\rho, z, t) = \tfrac{1}{2}\epsilon_{2pq}\{\epsilon_{2jn}\epsilon_{pim} + \epsilon_{2in}\epsilon_{pjm}\}\,\rho\,\partial_q[(1/\rho)\,A_{mn}] + \tfrac{1}{2}\langle\mu_{ij}(\boldsymbol{X}) + \mu_{ji}(\boldsymbol{X})\rangle. \tag{B 3}$$

The derivation of (B 1), (B 2) is divided into four parts. The vector $\boldsymbol{B}(\rho, z, t)$ is (*a*) restricted to be $B(\rho, z, t)\,\boldsymbol{i}_\phi$ and (*b*) restricted to be $\nabla\times[\chi(\rho, z, t)\,\boldsymbol{i}_\phi]$. In each case $[\mathscr{E}_0]_\phi$ and $[\nabla\times\mathscr{E}_0]_\phi$ are considered separately. Superposition of the results then leads to (B 1), (B 2).

(*a*) The vector $\boldsymbol{B}$ is azimuthal ($\boldsymbol{B} = B\boldsymbol{i}_\phi$). From (A 2), $[\mathscr{E}_0]_\phi$ is given trivially by

$$-[\mathscr{E}_0]_\phi = \langle\mu_{22}(\boldsymbol{X})\rangle\,B. \tag{B 4}$$

Taking the ϕ-component of the curl of (A 2) gives

$$-[\nabla\times\mathscr{E}_0]_\phi = \partial_i[(1/\rho)\,A_{ij}\,\partial_j(\rho B)] - \epsilon_{2ij}\,\partial_i\{[\epsilon_{2kj}(1/\rho)\,A_{1k} - \langle\mu_{j2}(\boldsymbol{X})\rangle]\,B\}. \tag{B 5}$$

However (A 1) gives the identity

$$\begin{aligned}\epsilon_{2kj}(1/\rho)\,\alpha_{1k}(\boldsymbol{X}) - \mu_{j2}(\boldsymbol{X}) &= \epsilon_{2jk}\,\partial_n[\alpha_{kn}(\boldsymbol{X})] - \mu_{2j}(\boldsymbol{X}),\\ &= \tfrac{1}{2}\epsilon_{2jk}\,\rho\,\partial_n[(1/\rho)\,\alpha_{kn}(\boldsymbol{X})] - \tfrac{1}{2}\{\mu_{j2}(\boldsymbol{X}) + \mu_{2j}(\boldsymbol{X})\}.\end{aligned} \tag{B 6}$$

Hence (B 5) and (B 6) establish the identity (B 2).

(*b*) The vector $\boldsymbol{B}$ is meridional ($\boldsymbol{B} = \nabla\times\chi\boldsymbol{i}_\phi$). Substituting $B_i = \epsilon_{2ij}(1/\rho)\,\partial_j(\rho\chi)$ into (A 2), it is a simple matter to verify that

$$-[\mathscr{E}_0]_\phi = \partial_i[(1/\rho)\,A_{ij}\,\partial_j(\rho\chi)] - [\epsilon_{2jk}\,\partial_n(A_{kn}) - \langle\mu_{2j}(\boldsymbol{X})\rangle]\,\epsilon_{2jp}(1/\rho)\,\partial_p(\rho\chi). \tag{B 7}$$

Evidently, together with the identity (B 6), this expression for $[\mathscr{E}_0]_\phi$ gives (B 1). The final reduction of $[\nabla\times\mathscr{E}_0]_\phi$ is more tedious. Lengthy but routine algebraic manipulations eventually lead to

$$-[\nabla\times\mathscr{E}_0]_\phi = \epsilon_{2ij}\,\partial_i[R_{jk}\,\epsilon_{2kn}(1/\rho)\,\partial_n(\rho\chi)], \tag{B 8}$$

where
$$R_{ij} = -\epsilon_{2pi}\,\rho\,\partial_j((1/\rho)\,A_{p2}) - \epsilon_{ijp}\,\partial_q(A_{qp}) + \langle\mu_{ij}(\boldsymbol{X})\rangle. \tag{B 9}$$

The identities $\Gamma_{11} = R_{11}$ and $\Gamma_{33} = R_{33}$ are self evident. It is also straightforward to show, using (A 1), that $R_{13} - R_{31} = 0$. Hence it follows that

$$\Gamma_{31} = \Gamma_{13} = \tfrac{1}{2}(R_{13} + R_{31}) = R_{13} = R_{31}, \tag{B 10}$$

and so (B 8) establishes the identity (B.2).

Appendix C

An explicit correspondence is made with the work of Braginskiĭ (1964*a*, *b*), by setting

$$\boldsymbol{X}(\boldsymbol{x}, t) = \boldsymbol{x} + R^{-\frac{1}{2}}\boldsymbol{\eta}(R, \boldsymbol{x}, t) \quad [\langle\boldsymbol{\eta}\rangle = O(R^{-\frac{1}{2}})]. \tag{C 1}$$

Since $J(\boldsymbol{X}) = 1$, it follows that

$$a_{kk}(\boldsymbol{\eta}) = -R^{-\frac{1}{2}}T_{kk}(\boldsymbol{\eta}) - R^{-1}J(\boldsymbol{\eta}), \tag{C 2}$$

and hence $T_{ij}(\boldsymbol{X})$ may be expanded in the form

$$T_{ij}(\boldsymbol{X}) = \delta_{ij} - R^{-\frac{1}{2}} a_{ji}(\boldsymbol{\eta}) + R^{-1}[T_{ij}(\boldsymbol{\eta}) - T_{kk}(\boldsymbol{\eta})\,\delta_{ij}] - R^{-\frac{3}{2}} J(\boldsymbol{\eta})\,\delta_{ij}. \tag{C 3}$$

The magnetic field and velocity at $\boldsymbol{X}$ are given by

$$b_i^*(\boldsymbol{X}, t) = b_i(\boldsymbol{x}, t) + R^{-\frac{1}{2}} a_{im}(\boldsymbol{\eta})\, b_m(\boldsymbol{x}, t), \tag{C 4a}$$

$$u_i^*(\boldsymbol{X}, t) = u_i(\boldsymbol{x}, t) + R^{-\frac{1}{2}}\{(\partial\boldsymbol{\eta}/\partial t)_i + a_{im}(\boldsymbol{\eta})\, u_m(\boldsymbol{x}, t)\}. \tag{C 4b}$$

The magnetic field at $\boldsymbol{x}$ is now determined by expanding $\boldsymbol{b}^*$ as a Taylor series at the point $\boldsymbol{X}$ and leads to

$$\boldsymbol{b}^*(\boldsymbol{x}, t) = \boldsymbol{b}^*(\boldsymbol{X}, t) - R^{-\frac{1}{2}}[\boldsymbol{\eta}\cdot\nabla_X \boldsymbol{b}^*]\,(\boldsymbol{X}, t) + \tfrac{1}{2}R^{-1}[\boldsymbol{\eta}\cdot\nabla_X(\boldsymbol{\eta}\cdot\nabla_X \boldsymbol{b}^*) - (\boldsymbol{\eta}\cdot\nabla_X\boldsymbol{\eta})\cdot\nabla_X \boldsymbol{b}^*]\,(\boldsymbol{X}, t) + O(R^{-\frac{3}{2}}). \tag{C 5}$$

Thus with the help of (2.7) and (C 3), (C 5) becomes

$$b_i^*(\boldsymbol{x}, t) = b_i^*(\boldsymbol{X}, t) - R^{-\frac{1}{2}}\eta_k a_{ik}(\boldsymbol{b}^*) + \tfrac{1}{2}R^{-1}[\eta_k a_{ik}(\boldsymbol{h}) + \eta_l a_{kl}(\boldsymbol{\eta})\, a_{ik}(\boldsymbol{b}^*)] + O(R^{-\frac{3}{2}}), \tag{C 6}$$

where $$h_i(\boldsymbol{x}, t) = \eta_l a_{il}(\boldsymbol{b}^*).$$

For the particular case where

$$\boldsymbol{b}(\boldsymbol{x}, t) = B(\rho, z, t)\,\boldsymbol{i}_\phi + R^{-1}\boldsymbol{b}_{\mathrm{eM}}(\rho, z, t) + R^{-\frac{3}{2}}\boldsymbol{b}'(\boldsymbol{x}, t), \tag{C 7a}$$

$$\boldsymbol{u}(\boldsymbol{x}, t) = U(\rho, z, t)\,\boldsymbol{i}_\phi + R^{-1}\boldsymbol{u}_{\mathrm{eM}}(\rho, z, t), \tag{C 7b}$$

it can be shown from (C 6) that the magnetic field and velocity at $\boldsymbol{x}$ may be expanded in the form

$$\boldsymbol{b}^*(\boldsymbol{x}, t) = \boldsymbol{b}^{(0)}(\boldsymbol{x}, t) + R^{-\frac{1}{2}}\boldsymbol{b}^{(1)}(\boldsymbol{x}, t) + R^{-1}\{\boldsymbol{b}^{(2)}(\boldsymbol{x}, t) + \boldsymbol{b}_{\mathrm{eM}}(\rho, z, t)\} + O(R^{-\frac{3}{2}}), \tag{C 8a}$$

$$\boldsymbol{u}^*(\boldsymbol{x}, t) = \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) + R^{-\frac{1}{2}}\boldsymbol{u}^{(1)}(\boldsymbol{x}, t) + R^{-1}\{\boldsymbol{u}^{(2)}(\boldsymbol{x}, t) + \boldsymbol{u}_{\mathrm{eM}}(\rho, z, t)\} + O(R^{-\frac{3}{2}}), \tag{C 8b}$$

where

$$\left.\begin{aligned}
&\boldsymbol{b}^{(0)}(\boldsymbol{x}, t) = B(\rho, z, t)\,\boldsymbol{i}_\phi, \quad \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) = U(\rho, z, t)\,\boldsymbol{i}_\phi,\\
&\boldsymbol{b}_{\mathrm{M}}^{(1)}(\boldsymbol{x}, t) = \frac{B}{\rho}\frac{\partial_1}{\partial\phi}\boldsymbol{\eta}_{\mathrm{M}}, \quad \boldsymbol{u}_{\mathrm{M}}^{(1)}(\boldsymbol{x}, t) = \left(\frac{\partial}{\partial t} + \frac{U}{\rho}\frac{\partial_1}{\partial\phi}\right)\boldsymbol{\eta}_{\mathrm{M}},\\
&\boldsymbol{b}_{\mathrm{M}}^{(2)}(\boldsymbol{x}, t) = \left\{-\nabla\cdot\left(\boldsymbol{\eta}\frac{B}{\rho}\frac{\partial\eta_\rho}{\partial\phi}\right) + \frac{B}{\rho}\frac{\partial}{\partial\phi}\left(\frac{1}{2\rho}\eta_\phi^2\right),\ 0,\ -\nabla\cdot\left(\boldsymbol{\eta}\frac{B}{\rho}\frac{\partial\eta_z}{\partial\phi}\right)\right\},\\
&\boldsymbol{u}_{\mathrm{M}}^{(2)}(\boldsymbol{x}, t) = \left\{-\nabla\cdot\left[\boldsymbol{\eta}\left(\frac{\partial}{\partial t} + \frac{U}{\rho}\frac{\partial}{\partial\phi}\right)\eta_\rho\right] + \left(\frac{\partial}{\partial t} + \frac{U}{\rho}\frac{\partial}{\partial\phi}\right)\left(\frac{1}{2\rho}\eta_\phi^2\right),\ 0,\right.\\
&\qquad\qquad\left.-\nabla\cdot\left[\boldsymbol{\eta}\left(\frac{\partial}{\partial t} + \frac{U}{\rho}\frac{\partial}{\partial\phi}\right)\eta_z\right]\right\},
\end{aligned}\right\} \tag{C 9}$$

the suffix $_{\mathrm{M}}$ denotes meridional components of the vector and $\partial_1/\partial\phi$ is the derivative with respect to ϕ: unit vectors are not differentiated.

In Braginskiĭ's (1964*a*) analysis where the time scale is order R, $\boldsymbol{u}^{(0)}$ is the mean axisymmetric azimuthal velocity, $R^{-\frac{1}{2}}\boldsymbol{u}^{(1)}$ is the fluctuating velocity, $R^{-1}(\langle\boldsymbol{u}_{\mathrm{M}}^{(2)}\rangle + \boldsymbol{u}_{\mathrm{eM}})$ is the mean meridional velocity, and $R^{-1}\boldsymbol{u}_{\mathrm{eM}}$ is the 'effective' meridional velocity. These identities are only correct to lowest order. A similar correspondence can be made for the magnetic field.

The mean quantity $\Gamma_{22}(\rho, z, t)$ is now considered. From (2.30*b*) and (2.45) it follows that

$$\Gamma_{22} = \epsilon_{2ij}\langle T_{pi}(\boldsymbol{X})\,\partial_2[T_{pj}(\boldsymbol{X})]\rangle, \tag{C 10}$$

and substituting the value of $T_{ij}(\boldsymbol{X})$ given by (C 3), (C 10) becomes

$$R\Gamma_{22} = \frac{2}{\rho}\left\langle a_{3p}(\boldsymbol{\eta})\frac{\partial}{\partial\phi}[a_{1p}(\boldsymbol{\eta})]\right\rangle + O(R^{-\frac{1}{2}}). \tag{C 11}$$

With the help of the definition (1.26) and the identity (C 2), this expression becomes

$$R\Gamma_{22} = \frac{2}{\rho^3}\left\langle\frac{\partial\eta_z}{\partial\phi}\eta_\rho + \frac{\partial\eta_z}{\partial\phi}\frac{\partial^2\eta_\rho}{\partial\phi^2}\right\rangle + \frac{2}{\rho}\left\langle\frac{\partial^2\eta_\rho}{\partial\rho\,\partial\phi}\frac{\partial\eta_z}{\partial\rho} + \frac{\partial^2\eta_\rho}{\partial z\,\partial\phi}\frac{\partial\eta_z}{\partial z} + \frac{1}{\rho}\frac{\partial\eta_z}{\partial\phi}\frac{\partial\eta_z}{\partial z}\right\rangle + O(R^{-\frac{1}{2}}). \quad \text{(C 12)}$$

Together with the expression for $(1/U)\,\boldsymbol{u}_{\mathrm{M}}^{(1)}$ given by (C 9) it follows that $R\Gamma_{22}$ is the Γ given by Braginskiĭ (1964*a*, eqn (3.21)).

The correspondence with Braginskiĭ (1964*a*, eqn (3.20)) is completed by setting Γ_{ij} (unless $i = j = 2$), $A_{ij} - \delta_{ij}$ and the mean of products of fluctuating quantities in (2.44), (2.45) equal to zero: χ is supposed order R^{-1}.

Finally the above arguments can be extended so that a correspondence can be obtained with Braginskiĭ (1964*b*, eqn (2.23)).

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